Handout 16: The bottleneck in turbulence

At large wavenumbers k, the energy spectrum

$$E(k) = \epsilon^{2/3}\, k^{5/3}\, f(k) \tag{1}$$

is expected to have a viscous cutoff that is described by the function $f(k)$. However, $f(k)$ decreases not necessarily monotonically with k.

1 Navier–Stokes equation in Fourier space

For an incompressible fluid, the Fourier-transformed Navier–Stokes equation for $\hat{\boldsymbol{u}}_{\boldsymbol{k}}(t)$ can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\boldsymbol{u}}_{\boldsymbol{k}} = -\mathrm{i}\boldsymbol{k}\hat{P}_{\boldsymbol{k}} - \sum_{\boldsymbol{k}=\boldsymbol{p}+\boldsymbol{q}} (\hat{\boldsymbol{u}}_{\boldsymbol{p}}\cdot \mathrm{i}\boldsymbol{q})\,\hat{\boldsymbol{u}}_{\boldsymbol{q}} - \nu\boldsymbol{k}^2\hat{\boldsymbol{u}}_{\boldsymbol{k}}. \tag{2}$$

The pressure satisfies a Poisson-type equation, so

$$\boldsymbol{k}^2 P_{\boldsymbol{k}} = \mathrm{i}\boldsymbol{k} \sum_{\boldsymbol{k}=\boldsymbol{p}+\boldsymbol{q}} (\hat{\boldsymbol{u}}_{\boldsymbol{p}}\cdot \mathrm{i}\boldsymbol{q})\,\hat{\boldsymbol{u}}_{\boldsymbol{q}} \tag{3}$$

and therefore

$$(\mathrm{i}\boldsymbol{k}P_{\boldsymbol{k}})_i = -\frac{k_i k_j}{k^2} \sum_{\boldsymbol{k}=\boldsymbol{p}+\boldsymbol{q}} (\hat{\boldsymbol{u}}_{\boldsymbol{p}}\cdot \mathrm{i}\boldsymbol{q})\,\hat{\boldsymbol{u}}_{\boldsymbol{q}} \tag{4}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\hat{\boldsymbol{u}}}_{\boldsymbol{k}} = -\mathsf{P}(\boldsymbol{k}) \sum_{\boldsymbol{k}=\boldsymbol{p}+\boldsymbol{q}} (\hat{\boldsymbol{u}}_{\boldsymbol{p}}\cdot \mathrm{i}\boldsymbol{q})\,\hat{\boldsymbol{u}}_{\boldsymbol{q}} - \nu\boldsymbol{k}^2\hat{\boldsymbol{u}}_{\boldsymbol{k}}, \tag{5}$$

where $\mathsf{P}_{ij} = \delta_{ij} - k_i k_j/k^2$ is the *projection operator*, which projects out the non-solenoidal (irrotational) components.

An important point here is the fact that all nonlinear inactions proceed via *triads* in $\boldsymbol{k}$ space where $\boldsymbol{k} = \boldsymbol{p}+\boldsymbol{q}$. It turns out that viscosity suppresses those triads that reach deep into the dissipative subrange. This makes nonlinear energy transfer less efficient and can lead to a *pileup* of energy in the inertial range shortly before the dissipative subrange (Falkovich, 1994). This is referred to as the bottleneck effect. To understand why it has not been a prominent effect in wind tunnel and atmospheric turbulence, we have to realize that most observed spectra have been obtained using hot-wire velocimetry and the Taylor hypothesis. We thus have to understand the relation between 1-D and 3-D energy spectra.

2 Relation between 1-D and 3-D energy spectra

To derive the relation between the three-dimensional spectrum $E(k)$ and the total one-dimensional spectrum $E_{\rm 1D}(k) \equiv E_{\rm L}(k) + 2E_{\rm T}(k)$, we consider a periodic box of volume $V = L_x L_y L_z$ with a turbulent velocity field $\boldsymbol{u}(\boldsymbol{x})$, which has the Fourier transform

$$\hat{\boldsymbol{u}}(\boldsymbol{k}) = \frac{1}{\sqrt{(2\pi)^3 V}} \int_V e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \boldsymbol{u}(\boldsymbol{x})\,\mathrm{d}x^3, \tag{6}$$

with the inversion

$$\boldsymbol{u}(\boldsymbol{x}) = \sqrt{\frac{V}{(2\pi)^3}} \int e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \hat{\boldsymbol{u}}(\boldsymbol{k})\,\mathrm{d}k^3. \tag{7}$$

The one-dimensional kinetic energy spectrum is

$$E_{\rm 1D}(k_z) = 2\iint \frac{\langle|\hat{\boldsymbol{u}}(\boldsymbol{k})|^2\rangle}{2}\,\mathrm{d}k_x\,\mathrm{d}k_y, \qquad (k_z \geq 0), \tag{8}$$

where $\langle\cdot\rangle$ denotes an ensemble average, and $\boldsymbol{k}=(k_x,k_y,k_z)$. The factor 2 in Eq. (8) accounts for the fact that $E_{\rm 1D}$ does not distinguish between positive and negative k_z. Normalization of $E_{\rm 1D}(k_z)$ is such that

$$\int_0^\infty E_{\rm 1D}(k_z)\,{\rm d}k_z=\frac{u_{\rm rms}^2}{2}\equiv\frac{1}{V}\int_V\frac{\left\langle|\boldsymbol{u}(\boldsymbol{x})|^2\right\rangle}{2}\,{\rm d}x^3. \tag{9}$$

Equation (8) can also be written as the xy-average

$$E_{\rm 1D}(k_z)=\frac{1}{L_xL_y}\int\left\langle|\tilde{\boldsymbol{u}}(x,y,k_z)|^2\right\rangle\,{\rm d}x\,{\rm d}y \tag{10}$$

and is for homogeneous turbulence equal to $\left\langle|\tilde{\boldsymbol{u}}(x,y,k_z)|^2\right\rangle$ at any point (x,y).

The three-dimensional velocity energy spectrum is given by

$$E(k)\equiv\int_{4\pi}\frac{\left\langle|\hat{\boldsymbol{u}}(\boldsymbol{k})|^2\right\rangle}{2}k^2\,{\rm d}\Omega_k, \tag{11}$$

where ${\rm d}\Omega_k$ denotes the solid angle element in $\boldsymbol{k}$-space. $E(k)$ satisfies the relation

$$\int_0^\infty E(k)\,{\rm d}k=\frac{u_{\rm rms}^2}{2}. \tag{12}$$

If $\boldsymbol{u}$ is statistically isotropic in the sense that the ensemble average of the spectral energy of the velocity $\left\langle|\boldsymbol{u}(\boldsymbol{k})|^2\right\rangle$ is only a function of $k=|\boldsymbol{k}|$, then $E(k)$ becomes

$$E(k)=4\pi k^2\frac{\left\langle|\hat{\boldsymbol{u}}(\boldsymbol{k})|^2\right\rangle}{2}. \tag{13}$$

To evaluate $E_{\rm 1D}$ in this case, we introduce cylindrical coordinates (k_r,ϕ,k_z) in $\boldsymbol{k}$-space and write the double integral (8) in the form

$$\begin{aligned} E_{\rm 1D}(k_z) &= 2\int_0^\infty\frac{\left\langle|\hat{\boldsymbol{u}}(\boldsymbol{k})|^2\right\rangle}{2}\,2\pi k_r\,{\rm d}k_r \\ &= 4\pi\int_{k_z}^\infty\frac{\left\langle|\hat{\boldsymbol{u}}(\boldsymbol{k})|^2\right\rangle}{2}\,k\,{\rm d}k, \end{aligned} \tag{14}$$

since $k^2=k_r^2+k_z^2$, and therefore $k_r^2=k^2-k_z^2$. Comparing with Eq. (13), we see that

$$E_{\rm 1D}(k_z)=\int_{k_z}^\infty\frac{E(k)}{k}\,{\rm d}k, \tag{15}$$

the inversion of which gives

$$E(k)=-k\frac{{\rm d}E_{\rm 1D}(k)}{{\rm d}k}=-E_{\rm 1D}\frac{{\rm d}\ln E_{\rm 1D}(k)}{{\rm d}\ln k}\ . \tag{16}$$

References

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