Handout 17: Vorticity production and Karman–Howarth equation

In Handout 15, we have seen that turbulence is full of structures. It consists of vortex tubes, at least at small scales. They are *not* really produced by stretching, as one might have thought, but rather by rotational straining motion or shear. Vortices are thus modes similarly as you would make spaghetti.

1 Vorticity production

Mathematically, one can see that vorticity is being produced by the rate-of-strain matrix, $s_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$, because the vorticity equation

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = \boldsymbol{\omega}\cdot\boldsymbol{\nabla}\boldsymbol{u} + \nu\nabla^2\boldsymbol{\omega} \tag{1}$$

involves just the symmetric part (the antisymmetric part can be written as $a_{ij} = -\frac{1}{2}\epsilon_{ijk}\omega_k$, so $a_{ij}\omega_j = -\frac{1}{2}\epsilon_{ijk}\omega_j\omega_k = 0$). Thus, we have

$$\frac{\mathrm{D}\omega_i}{\mathrm{D}t} = s_{ij}\omega_j + \nu\nabla^2\omega_i. \tag{2}$$

It is therefore useful to study the principal axes of s_{ij} and to ask how the vorticity vector aligns itself in the strain field. Numerical simulations have shown conclusively that $\boldsymbol{\omega}$ aligns itself with the *intermediate* eigenvector of the rate-of-strain matrix (Vincent & Meneguzzi, 1991); see Figure 1 for a corresponding result in MHD, where it turns out that, by contrast, the magnetic field vector aligns itself with the direction of *shear*, which is at 45° angles with the directions of *both* stretching and compression (Brandenburg et al., 1995).

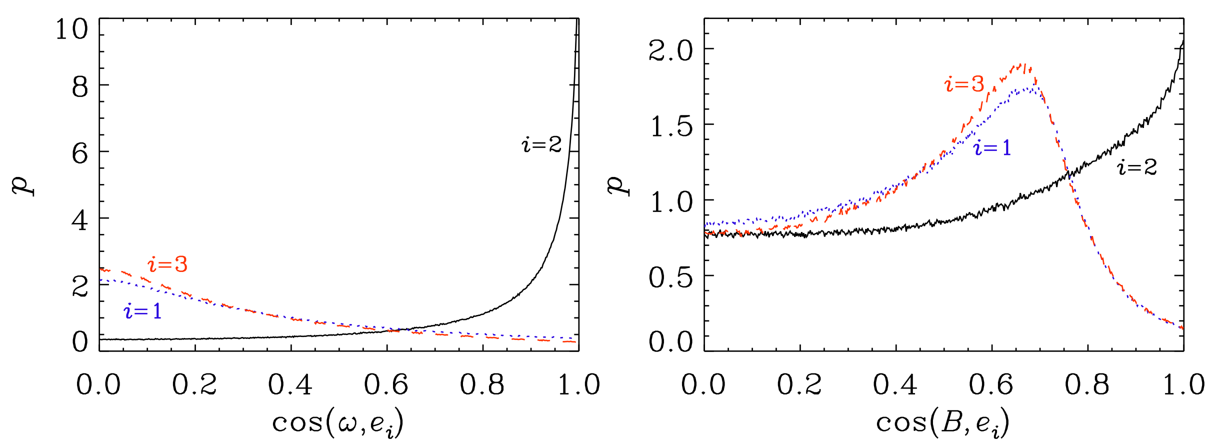

Figure 1: Alignment of $\boldsymbol{\omega}$ and $\boldsymbol{B}$ with the eigenvectors of the rate-of-strain tensor. Adapted from the Supplemental Material of Brandenburg et al. (2015).

The production rate of enstrophy is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \tfrac{1}{2}\boldsymbol{\omega}^2\right\rangle = \left\langle s_{ij}\omega_i\omega_j\right\rangle - \nu\left\langle(\boldsymbol{\nabla}\times\boldsymbol{\omega})^2\right\rangle. \tag{3}$$

(Incidently, a similar relation also holds for the magnetic field.) In a statistically steady state, $\langle s_{ij}\omega_i\omega_j\rangle$ must be positive.

2 Skewness

The term $\langle s_{ij}\omega_i\omega_j\rangle$ is cubic in the velocity derivatives, so it is not too surprising that one can also relate it to the cube of the velocity differences, $(\Delta v)^3$, at least for isotropic turbulence and in the limit of small separation r; see Davidson (2015) for details. The calculation shows that

$$\langle s_{ij}\omega_i\omega_j\rangle = -\tfrac{35}{2}\left[\left\langle(\Delta v)^3\right\rangle/r^3\right]_{r\to 0}. \tag{4}$$

The minus sign is here important. Many measurements of turbulence show that $\left\langle(\Delta v)^3\right\rangle$ is indeed negative. It means that the *probability density function* (PDF) of Δv is skewed to the left; see Figure 2 for an example of a PDF of Δv normalized by its rms value.

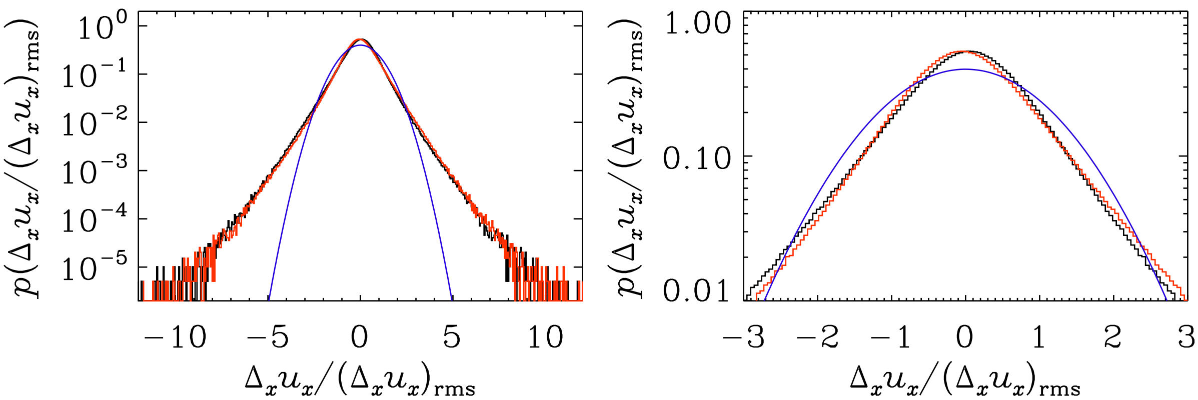

Figure 2: Histograms of $\Delta_x u_x$ (normalized by its rms value). The left panel shows the full range while the right one the center.

The shapes of PDFs is usually well characterized by its moments. The normalized third moment, $\left\langle(\Delta v)^3\right\rangle/\left\langle(\Delta v)^2\right\rangle^{3/2}$, is called the skewness, and, independently of Reynolds number, its value is found to be in the range from -0.5 to -0.4. The fourth moment, $\left\langle(\Delta v)^4\right\rangle/\left\langle(\Delta v)^2\right\rangle^2$, is called the kurtosis. Its value would be 3 for a Gaussian distribution, but turbulence is highly non-Gaussian with elevated wings of the distribution, so the kurtosis is usually much bigger that 3.

The skewed distribution of velocity difference is thus crucial for for the production and perhaps the very existence of vorticity! In fact, Kolmogorov found that it is related to the *energy dissipation rate* ϵ via

$$\left\langle(\Delta v)^3\right\rangle = -\frac{4}{5}\epsilon r \quad \text{for } r \text{ within the inertial range.} \tag{5}$$

where r is the separation between the two measurement points. This law is sometimes called the *four-fifths law* and it is an *exact* result. It follows as a special application of the Karman–Howarth equation, which will be explain next.

3 Karman–Howarth equation

The Karman–Howarth equation is a real-space equation for the two-point correlation function

$$Q_{ij}(\boldsymbol{r}) = \langle u_i(\boldsymbol{x})u_j(\boldsymbol{x}+\boldsymbol{r})\rangle. \tag{6}$$

It can also be written as $Q_{ij}(\boldsymbol{x},\boldsymbol{x}') = \langle u_i(\boldsymbol{x})u_j(\boldsymbol{x}')\rangle$ and is therefore often abbreviated as $\langle u_i u_j'\rangle$. The evolution equation for $Q_{ij}(\boldsymbol{r})$ will have triple correlations on the right-hand side, which are defined as

$$S_{ijk}(\boldsymbol{r}) = \langle u_i(\boldsymbol{x})u_j(\boldsymbol{x})u_k(\boldsymbol{x}+\boldsymbol{r})\rangle. \tag{7}$$

Of particular interest for the following are special case such as

$$Q_{xx}(r\hat{\boldsymbol{e}}_x) = u^2\, f(r), \tag{8}$$

$$Q_{yy}(r\hat{\boldsymbol{e}}_x) = u^2\, g(r), \tag{9}$$

which are also known as the longitudinal and lateral correlation functions. Likewise, we have

$$S_{xxx}(r\hat{\boldsymbol{e}}_x) = u^3\, K(r). \tag{10}$$

To derive the equation for Q_{ij}, one starts with the momentum equation and multiplies by $\boldsymbol{u}'$, and likewise the momentum equation for $\boldsymbol{u}'$, which is then multiplied by $\boldsymbol{u}$. Thus,

$$\frac{\partial u_i}{\partial t} = -\frac{\partial}{\partial x_k}\left(u_i u_k\right) - \frac{\partial}{\partial x_i}\left(p/\rho\right) + \nu\nabla^2_{\boldsymbol{x}} u_i, \tag{11}$$

$$\frac{\partial u_j'}{\partial t} = -\frac{\partial}{\partial x_k}\left(u_j' u_k'\right) - \frac{\partial}{\partial x_j}\left(p'/\rho\right) + \nu\nabla^2_{\boldsymbol{x}'} u_j'. \tag{12}$$

Multiplying the first of these by u_j' and the second one by u_i, and adding, we obtain

$$\frac{\partial}{\partial t}\langle u_i u_j'\rangle = -\left\langle u_i \frac{\partial}{\partial x_k'}\left(u_j' u_k'\right)\right\rangle - \left\langle u_j' \frac{\partial}{\partial x_k}\left(u_i u_k\right)\right\rangle + ... \tag{13}$$

To continue from here, one has to use a number of tricks in the manipulation of two-point correlation tensors, such as

- Averaging and differentiation commute,
- $\partial/\partial x_i$ and $\partial/\partial x_j'$ can be replaced by $-\partial/\partial r_i$ and $\partial/\partial r_j$,
- $\langle u_i u_j' u_k'\rangle(\boldsymbol{r}) = \langle u_j u_k u_i'\rangle(-\boldsymbol{r}) = -\langle u_j u_k u_i'\rangle(\boldsymbol{r})$
-

With this one obtains

$$\frac{\partial}{\partial t} Q_{ij} = \frac{\partial}{\partial r_k}\left(S_{ikj} + S_{jki}\right) + 2\nu\nabla^2 Q_{ij}. \tag{14}$$

Of particular interest is the connection between f and K, which turns out to be

$$\frac{\partial}{\partial t}\left[u^2 f(r,t)\right] = \frac{1}{r^4}\frac{\partial}{\partial r}\left[r^4 u^3 K(r)\right] + 2\nu\frac{1}{r^4}\frac{\partial}{\partial r}\left[r^4 u^2 f'(r)\right] \tag{15}$$

where $f'(r)$ is the r derivative. This is a special form of the Karman–Howarth equation. Applying it to turbulence leads to the following relation

$$r^4\frac{\partial}{\partial t}\left(\epsilon^{2/3} r^{2/3}\right) \sim r^4 (u/\ell)\epsilon^{2/3} r^{2/3} \sim r^4 \epsilon (r/\ell)^{2/3}. \tag{16}$$

References

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