Handout 18: Decaying turbulence

In the absence of forcing, turbulence can only decay. The energy dissipation rate still plays an important role, but it is no longer constant, but it determines nor the decay of the mean squared velocity, specifically $\frac{1}{2}\langle \boldsymbol{u}^2\rangle \equiv \mathcal{E}(t)$. We use here $\mathcal{E}(t)$ to denote the mean energy density and to distinguish it from the energy spectrum $E(k,t)$. The two are of course related via

$$\mathcal{E}(t) = \int_0^\infty E(k,t)\,\mathrm{d}k. \tag{1}$$

In decaying turbulence, both functions are decaying, and this is governed just by ϵ, so we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) = -\epsilon(t). \tag{2}$$

Using dimensional arguments, we have $\epsilon \sim U^3/\xi$, where U is the typical velocity and ξ some typical length scale of the turbulence. Both are time-dependent; U can be related to $\mathcal{E}(t) \sim U^2$ and $\xi(t) \propto t^q$ is as yet undetermined, but in decaying turbulence, the small scales will die out, so only large eddies remain, so we expect $\xi(t)$ to grow, so $q > 0$. Once we know q, we can proceed and write

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) = -\mathcal{E}^{3/2}t^{-q}. \tag{3}$$

This can be integrated

$$\int \mathcal{E}^{-3/2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) = -\int t^{-q}\mathrm{d}t, \tag{4}$$

so $\mathcal{E}^{-1/2} \sim t^{1-q}$ or

$$\mathcal{E} \propto t^{-p}, \quad \text{with } p = 2(1-q). \tag{5}$$

1 Relation to conservation laws

Conflicting results about the decay can be found in the literature. The results depend on the governing physical processes involved and the initial conditions.

Kolmogorov made predictions under the assumption that the so-called Loitsianskii[1] integral is conserved. This integral is related to the angular momentum, $\boldsymbol{x} \times \boldsymbol{u}$. The actual Loitsianskii integral is defined in terms of a two-point correlation function as

$$\mathcal{L} = \int \boldsymbol{r}^2 \langle \boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r})\rangle\, d^3\boldsymbol{r}, \tag{6}$$

so it has dimensions

$$\mathcal{L} \propto \ell^5 u_\ell^2 \propto L^7 T^{-2}. \tag{7}$$

This suggests that $q = 2/7$. With such an assumption, one finds from Equation (5)

$$p = 2(1-2/7) = 2 \times 5/7 = 10/7 \approx 1.43. \tag{8}$$

This did not agree too well with the available experiments.

Later, Saffman (1967) proposed that the linear momentum might be "more conserved". The debate is ongoing (Davidson, 2000), but according to Saffman the relevant integral is

$$\mathcal{S} = \int \langle \boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r})\rangle\, d^3\boldsymbol{r}, \tag{9}$$

which has the dimension L^5T^{-2}, so $q = 2/5$ and therefore

$$p = 2(1-2/5) = 2 \times 3/5 = 6/5 \approx 1.2. \tag{10}$$

[1]Different spellings can be found, depending on who translated first into which language. Alternative spellings include: Loitsyanskii, Loitsiansky, and Loitsyansky

Table 1: Scaling exponents and relation to physical invariants and their dimensions.

α	p	q	inv.	dim.
4	$10/7 \approx 1.43$	$2/7 \approx 0.286$	$\mathcal{L}$	$[x]^7[t]^{-2}$
3	$8/6 \approx 1.33$	$2/6 \approx 0.333$		
2	$6/5 = 1.20$	$2/5 = 0.400$		
1	$4/4 = 1.00$	$2/4 = 0.500$	$\langle \boldsymbol{A}_{\rm 2D}^2 \rangle$	$[x]^4[t]^{-2}$
0	$2/3 \approx 0.67$	$2/3 \approx 0.667$	$\langle \boldsymbol{A} \cdot \boldsymbol{B} \rangle$	$[x]^3[t]^{-2}$
-1	$0/2 = 0.00$	$2/1 = 1.000$		

2 Relation to initial conditions

A completely different idea is to assume a connetion with the initial spectrum, so let us write

$$E(k,t) \propto k^\alpha, \tag{11}$$

up to some cutoff scale, so we better write

$$E(k,t) \propto k^\alpha \psi k \xi(t), \tag{12}$$

Integrating over k yields

$$\mathcal{E}(t) = \int_0^\infty E(k,t)\,\mathrm{d}k = \xi^{-(\alpha+1)} \int_0^\infty (k\xi)^\alpha \psi(k\xi,t)\,\mathrm{d}k\xi \tag{13}$$

so

$$\mathcal{E}(t) \propto \xi^{-(\alpha+1)}. \tag{14}$$

The Navier-Stokes equations are invariant under rescaling, $x \to \tilde{x}\ell$ and $t \to \tilde{t}\ell^{1/q}$, which implies corresponding rescalings for velocity $u \to \tilde{t}\ell^{1-1/q}$ and viscosity $\nu \to \tilde{\nu}\ell^{2-1/q}$. However, ψ should be universal, but since it has dimensions,

$$[\psi] = [E][L]^\alpha = L^{3+\alpha}T^{-2}, \tag{15}$$

we can determine q; see Table 1 for a summary and Figure 1 for numerical results for α close to 4.

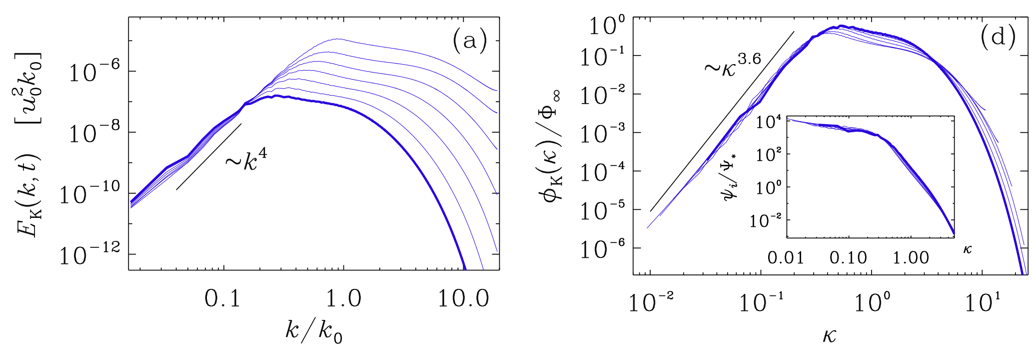

Figure 1: Spectra at different times (left) and collapsed spectra (right).

References

Davidson, P. A., "Was Loitsyansky correct? A review of the arguments," *J. Turb.* **1**, N6 (2000).

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Saffman, P. G., "Note on decay of homogeneous turbulence," *Phys. Fluids* **10**, 1349-1349 (1967).

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