Handout 19: Inverse Cascades and Inverse Transfer

Inverse cascades are not that uncommon, but they are usually always related to some extra conservation laws in turbulence. Magnetic helicity in 3-D MHD is perhaps the most famous one, but in 2-D MHD we have the conservation of $\langle \boldsymbol{A}^2 \rangle$ and in 2-D HD we have the conservation of $\langle \boldsymbol{\omega}^2 \rangle$. In closed or periodic domains we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \boldsymbol{\omega}^2 \rangle = -2\nu \langle (\boldsymbol{\nabla} \times \boldsymbol{\omega})^2 \rangle \quad \text{(in 2-D HD)}, \tag{1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \boldsymbol{A}^2 \rangle = -2\eta \langle \boldsymbol{B}^2 \rangle \quad \text{(in 2-D MHD)}, \tag{2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \boldsymbol{A} \cdot \boldsymbol{B} \rangle = -2\eta \langle \boldsymbol{J} \cdot \boldsymbol{B} \rangle \quad \text{(in 3-D MHD)}, \tag{3}$$

The consequences can be dramatic. Below some examples.

1 2-D Hydrodynamic Turbulence

The possibility of an inverse cascase in 2-D hydrodynamic turbulence was demonstrated conclusively by Frisch & Sulem (1984); see Figure 1 the spectra from their paper. In this case, energy was injected at *intermediate* lenth scales. This is important, because otherwise there is no room for the inverse cascase to move energy to larger length scales.

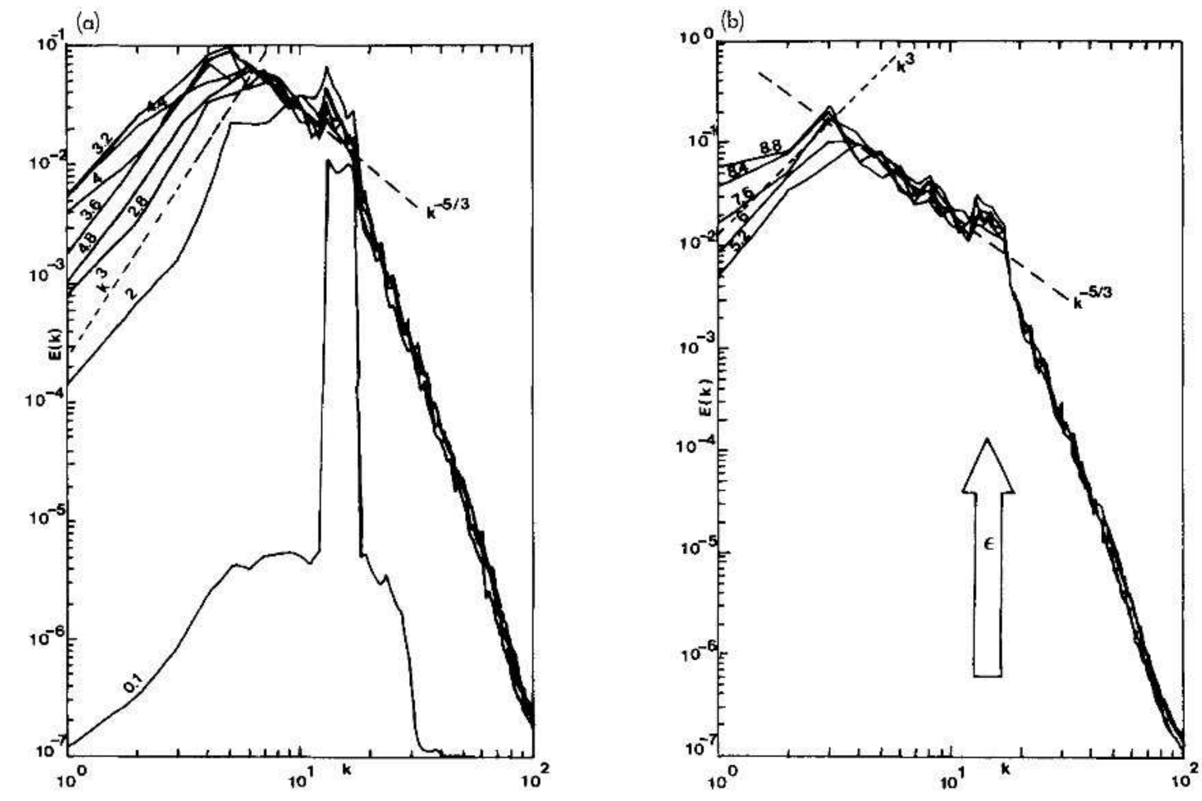

Figure 1: Forced 2-D hydrodynamic turbulence (Frisch & Sulem, 1984).

2 3-D MHD Turbulence

The case of decaying turbulence is perhaps most dramatic, because we actually see a real *increase* of spectral energy at small wavenumbers; see Figure 2. To understand what happens, we first need to explain the realizability condition, which means that magnetic helicity can only be realized in a field up to a certain value.

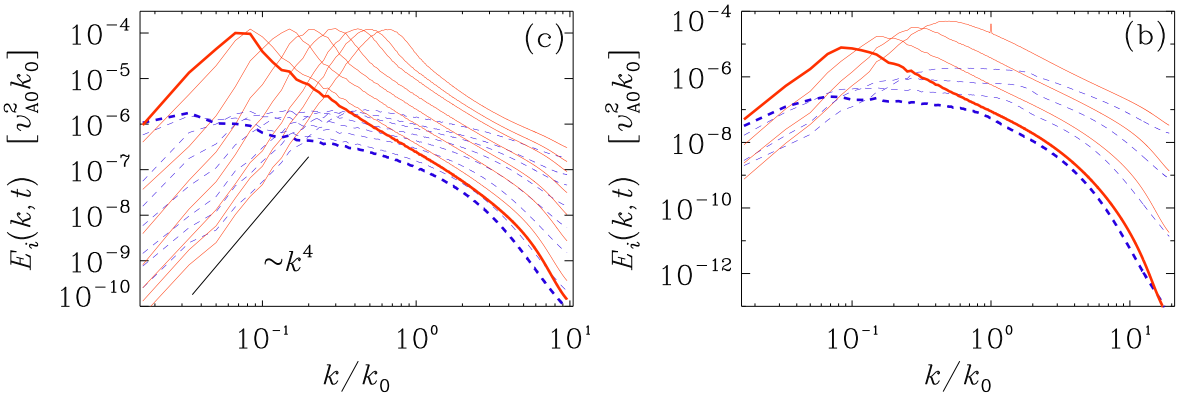

Figure 2: Comparsion of spectra of decaying MHD turbulence with helicity (left) and without (right).

3 Realizability Condition

Magnetic energy and helicity spectra are usually calculated as

$$M_k = \frac{1}{2}\int_{k\text{-shell}} \boldsymbol{B}_{\boldsymbol{k}}^* \cdot \boldsymbol{B}_{\boldsymbol{k}}\, k^2\, \mathrm{d}\Omega_k, \tag{4}$$

$$H_k = \frac{1}{2}\int_{k\text{-shell}} (\boldsymbol{A}_{\boldsymbol{k}}^* \cdot \boldsymbol{B}_{\boldsymbol{k}} + \boldsymbol{A}_{\boldsymbol{k}} \cdot \boldsymbol{B}_{\boldsymbol{k}}^*)\, k^2\, \mathrm{d}\Omega_{\boldsymbol{k}}, \tag{5}$$

where $\mathrm{d}\Omega_{\boldsymbol{k}}$ is the solid angle element in Fourier space, $\boldsymbol{B}_{\boldsymbol{k}} = \mathrm{i}\boldsymbol{k} \times \boldsymbol{A}_{\boldsymbol{k}}$ is the Fourier transform of the magnetic field, and $\boldsymbol{A}_{\boldsymbol{k}}$ is the Fourier transform of the vectors potential. These spectra are normalized such that

$$\int_0^\infty H_k\, \mathrm{d}k = \langle \boldsymbol{A} \cdot \boldsymbol{B} \rangle V \equiv H, \tag{6}$$

$$\int_0^\infty M_k\, \mathrm{d}k = \langle \tfrac{1}{2}\boldsymbol{B}^2 \rangle V \equiv M, \tag{7}$$

where H and M are magnetic helicity and magnetic energy, respectively, and angular brackets denote volume averages.

It is convenient to decompose the Fourier transformed magnetic vector potential, $\boldsymbol{A}_{\boldsymbol{k}}$, into a longitudinal component, $\boldsymbol{h}^{\parallel}$, and eigenfunctions $\boldsymbol{h}^{\pm}$ of the curl operator. This decomposition has been used in studies of turbulence Waleffe (1993). We have seen such functions as Beltrami waves, but they can be oriented in arbitrary directions.

$$\boldsymbol{h}(\boldsymbol{k}) = \mathsf{R} \cdot \boldsymbol{h}(\boldsymbol{k})^{(\mathrm{nohel})} \quad \text{with} \quad \mathsf{R}_{ij} = \frac{\delta_{ij} - \mathrm{i}\sigma\epsilon_{ijk}\hat{k}}{\sqrt{1+\sigma^2}}, \tag{8}$$

where the parameter σ characterizes the fractional helicity of $\boldsymbol{f}$, and

$$\boldsymbol{h}(\boldsymbol{k})^{(\mathrm{nohel})} = (\boldsymbol{k} \times \hat{\boldsymbol{e}}) / \sqrt{\boldsymbol{k}^2 - (\boldsymbol{k} \cdot \hat{\boldsymbol{e}})^2}, \tag{9}$$

is a non-helical forcing function. Here $\hat{\boldsymbol{e}}$ is an arbitrary unit vector not aligned with $\boldsymbol{k}$, $\hat{\boldsymbol{k}}$ is the unit vector along $\boldsymbol{k}$, and $|\boldsymbol{h}|^2 = 1$.

Using this decomposition we can write the Fourier transformed magnetic vector potential as

$$\boldsymbol{A_k} = a_{\boldsymbol{k}}^{+}\boldsymbol{h}_{\boldsymbol{k}}^{+} + a_{\boldsymbol{k}}^{-}\boldsymbol{h}_{\boldsymbol{k}}^{-} + a_{\boldsymbol{k}}^{\parallel}\boldsymbol{h}_{\boldsymbol{k}}^{\parallel}, \tag{10}$$

with

$$\mathrm{i}\boldsymbol{k} \times \boldsymbol{h}_{\boldsymbol{k}}^{\pm} = \pm k\boldsymbol{h}_{\boldsymbol{k}}^{\pm}, \qquad k = |\boldsymbol{k}|, \tag{11}$$

and

$$\langle \boldsymbol{h}_{\boldsymbol{k}}^{+*} \cdot \boldsymbol{h}_{\boldsymbol{k}}^{+} \rangle = \langle \boldsymbol{h}_{\boldsymbol{k}}^{-*} \cdot \boldsymbol{h}_{\boldsymbol{k}}^{-} \rangle = \langle \boldsymbol{h}_{\boldsymbol{k}}^{\parallel *} \cdot \boldsymbol{h}_{\boldsymbol{k}}^{\parallel} \rangle = 1, \tag{12}$$

where asterisks denote the complex conjugate, and angular brackets denote, as usual, volume averages. The longitudinal part $a_{\boldsymbol{k}}^{\parallel}\boldsymbol{h}_{\boldsymbol{k}}^{\parallel}$ is parallel to $\boldsymbol{k}$ and vanishes after taking the curl to calculate the magnetic field. In the Coulomb gauge, $\nabla \cdot \boldsymbol{A} = 0$, the longitudinal component vanishes altogether.

The (complex) coefficients $a_{\boldsymbol{k}}^{\pm}(t)$ depend on $\boldsymbol{k}$ and t, while the eigenfunctions $\boldsymbol{h}_{\boldsymbol{k}}^{\pm}$, which form an orthonormal set, depend only on $\boldsymbol{k}$ and are given by

$$\boldsymbol{h}_{\boldsymbol{k}}^{\pm} = \frac{1}{\sqrt{2}} \frac{\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{e}) \mp \mathrm{i}k(\boldsymbol{k} \times \boldsymbol{e})}{k^2\sqrt{1 - (\boldsymbol{k} \cdot \boldsymbol{e})^2/k^2}}, \tag{13}$$

where $\boldsymbol{e}$ is an arbitrary unit vector that is not parallel to $\boldsymbol{k}$. With these preparations we can write the magnetic helicity and energy spectra in the form

$$H_k = k(|a_k^+|^2 - |a_k^-|^2)V, \tag{14}$$

$$M_k = \tfrac{1}{2}k^2(|a_k^+|^2 + |a_k^-|^2)V, \tag{15}$$

where V is the volume of integration. (Here again the factor μ_0^{-1} is ignored in the definition of the magnetic energy.) From Equations (14) and (15) one sees immediately that

$$\tfrac{1}{2}k|H_k| \leq M_k, \tag{16}$$

which is also known as the *realizability condition*. A fully helical field has therefore $M_k = \pm\frac{1}{2}kH_k$.

For further reference we now define power spectra of those components of the field that are either right or left handed, i.e.

$$H_k^{\pm} = \pm k|a_k^{\pm}|^2 V, \quad M_k^{\pm} = \tfrac{1}{2}k^2|a_k^{\pm}|^2 V. \tag{17}$$

Thus, we have $H_k = H_k^+ + H_k^-$ and $M_k = M_k^+ + M_k^-$. Note that $H_k^{\pm}$ and $M_k^{\pm}$ can be calculated without explicit decomposition into right and left handed field components using

$$H_k^{\pm} = \tfrac{1}{2}(H_k \pm 2k^{-1}M_k), \quad M_k^{\pm} = \tfrac{1}{2}(M_k \pm \tfrac{1}{2}kH_k). \tag{18}$$

This method is significantly simpler than invoking explicitly the decomposition in terms of $a_{\boldsymbol{k}}^{\pm}\boldsymbol{h}_{\boldsymbol{k}}^{\pm}$.

In Section **??** plots of $M_k^{\pm}$ will be shown and discussed in connection with turbulence simulations. Here the turbulence is driven with a helical forcing function proportional to $\boldsymbol{h}_{\boldsymbol{k}}^{+}$; see Equation (13).

4 Argument for inverse cascade

The occurrence of an inverse cascade can be understood as the result of two waves (wavenumbers $\boldsymbol{p}$ and $\boldsymbol{q}$) interacting with each other to produce a wave of wavenumber $\boldsymbol{k}$. The following argument is due to Frisch et al. (1975). Assuming that during this process magnetic energy is conserved together with magnetic helicity, we have

$$M_p + M_q = M_k, \tag{19}$$

$$|H_p| + |H_q| = |H_k|, \tag{20}$$

where we are assuming that only helicity of one sign is involved. Suppose the initial field is fully helical and has the same sign of magnetic helicity at all scales, we have

$$2M_p = p|H_p| \quad \text{and} \quad 2M_q = q|H_q|, \tag{21}$$

and so Equation (19) yields

$$p|H_p| + q|H_q| = 2M_k \geq k|H_k|, \tag{22}$$

where the last inequality is just the realizability condition (16) applied to the target wavenumber $\boldsymbol{k}$ after the interaction. Using Equation (20) in Equation (22) we have

$$p|H_p| + q|H_q| \geq k(|H_p| + |H_q|). \tag{23}$$

In other words, the target wavevector $\boldsymbol{k}$ after the interaction of wavenumbers $\boldsymbol{p}$ and $\boldsymbol{q}$ satisfies

$$k \leq \frac{p|H_p| + q|H_q|}{|H_p| + |H_q|}. \tag{24}$$

The expression on the right hand side of Equation (24) is a weighted mean of p and q and thus satisfies

$$\min(p,q) \leq \frac{p|H_p| + q|H_q|}{|H_p| + |H_q|} \leq \max(p,q), \tag{25}$$

and therefore

$$k \leq \max(p,q). \tag{26}$$

In the special case where $p = q$, we have $k \leq p = q$, so the target wavenumber after interaction is always less or equal to the initial wavenumbers. In other words, wave interactions tend to transfer magnetic energy to smaller wavenumbers, i.e. to larger scale. This corresponds to an inverse cascade. The realizability condition, $\frac{1}{2}k|H_k| \leq M_k$, was the most important ingredient in this argument.

References

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