

# Handout 1: Thermal Instability

The Rayleigh-Benard instability is usually characterized as a thermal instability. However, the restoring force in that case is gravity, and so it is in that sense a dynamical instability. True thermal instabilities have a restoring term in the temperature equation. The thermal instability is technically simple and can be discussed even in one dimension. An influential paper on thermal instability was written by Field (1965). Earlier work by Parker (1953) gave a less stringent stability criterion, because he assumed constant density (the isochoric case). This handout gives the basic derivation.

## 1 Cooling function and entropy equation

The energy equation can be formulated as an evolution equation for the specific entropy,

$$\rho T \frac{Ds}{Dt} = -\rho \mathcal{L} + \nabla \cdot (K \nabla T), \quad (1)$$

where  $-\mathcal{L} = \rho \Lambda - \Gamma$  is the difference between heating and cooling (per unit mass, and  $-\rho \mathcal{L}$  is per unit volume). Note that the cooling per unit volume is proportional to  $\rho^2$ , reflecting the fact the cooling is accomplished by the collision of *two* partners. However, in the present context, we will work with temperature and density, so we rewrite this using

$$DS/c_p = \frac{1}{\gamma} D \ln P - D \ln \rho \quad (\text{entropy equation}) \quad (2)$$

and

$$D \ln P = D \ln T + D \ln \rho \quad (\text{perfect gas equation}). \quad (3)$$

Thus, we have

$$DS/c_p = \frac{1}{\gamma} D \ln T - \left(1 - \frac{1}{\gamma}\right) D \ln \rho. \quad (4)$$

We can then write the energy equation in the form

$$c_p \rho T \left[ \frac{1}{\gamma} \frac{D \ln T}{Dt} - \left(1 - \frac{1}{\gamma}\right) \frac{D \ln \rho}{Dt} \right] = -\rho \mathcal{L} + \nabla \cdot (K \nabla T). \quad (5)$$

Using  $\gamma = c_p/c_v$ , we have

$$c_v \rho \frac{DT}{Dt} - c_p T \left(1 - \frac{1}{\gamma}\right) \frac{D\rho}{Dt} = -\rho \mathcal{L} + \nabla \cdot (K \nabla T). \quad (6)$$

## 2 Linearized cooling function

Let us linearize  $\mathcal{L}$  (i.e., Taylor expand to first order) about  $\rho = \rho_0$  and  $T = T_0$ ,

$$\mathcal{L}(\rho, T) \approx \mathcal{L}_0 + \left(\frac{\partial \mathcal{L}}{\partial \rho}\right)_T (\rho - \rho_0) + \left(\frac{\partial \mathcal{L}}{\partial T}\right)_\rho (T - T_0), \quad (7)$$

or, in abbreviated form using  $\mathcal{L}_\rho = (\partial \mathcal{L} / \partial \rho)_T$ ,  $\mathcal{L}_T = (\partial \mathcal{L} / \partial T)_\rho$ , we have

$$\mathcal{L}(\rho, T) \approx \mathcal{L}_0 + \mathcal{L}_\rho (\rho - \rho_0) + \mathcal{L}_T (T - T_0). \quad (8)$$

If  $\rho_0$  and  $T_0$  refer to an equilibrium state, then  $\mathcal{L}_0 = 0$ .

### 3 Isochoric case ( $\rho = \text{const}$ )

If  $\rho = \text{const}$ , i.e.,  $D\rho/Dt = 0$  (isochoric case), we can write

$$c_v \rho \frac{DT}{Dt} = -\rho \mathcal{L}_0 - \rho \mathcal{L}_T (T - T_0) + \nabla \cdot (K \nabla T). \quad (9)$$

There is an equilibrium solution with  $T = T_0$ ,  $\rho = \rho_0$ , and  $\mathbf{u} = 0$ , so  $D/Dt = \partial/\partial t$  provided  $\mathcal{L}_0 = 0$ . We can then linearize around this state, so we write  $T = T_0 + T_1$ , i.e.,

$$c_v \rho_0 \frac{\partial T_1}{\partial t} = -\rho_0 \mathcal{L}_T T_1 + K \nabla^2 T_1. \quad (10)$$

where we have assumed  $K = \text{const}$ . Let us now seek solutions of the form  $T_1 = \hat{T}_1 e^{\sigma t + i\mathbf{k} \cdot \mathbf{x}}$ , which leads to the following characteristic equation (or eigenvalue problem),

$$c_v \rho_0 \sigma = -\rho_0 \mathcal{L}_T - K k^2. \quad (11)$$

where  $\sigma$  is the eigenvalue. All eigenvalues are positive and instability occurs when  $\mathcal{L}_T$  is sufficiently negative, i.e.,  $-\mathcal{L}_T - K k^2 > 0$ , or

$$\rho_0 \mathcal{L}_T < -K k^2 \quad (\text{for instability}). \quad (12)$$

This shows that a *necessary* condition for instability is

$$\mathcal{L}_T \equiv \left( \frac{\partial \mathcal{L}}{\partial T} \right)_\rho < 0 \quad (\text{for instability}). \quad (13)$$

A negative  $\mathcal{L}_T$  means that cooling decreases as it gets hotter, so no surprise that this leads to instability. Note, however, that *thermal diffusion has a stabilizing effect*.

Of course, in a compressible gas there is no reason that  $\rho$  should be constant. Another possibility is that the pressure stays constant. In that case, if it does get hotter, the density decreases, so  $\rho \mathcal{L}$  decreases further, making the system even more unstable. To treat this more general case, we have to invoke the hydrodynamic equations, which will be done in the next section.

### 4 The non-isochoric case

In the non-isochoric case we also consider the consider the linearized continuity equation,

$$\frac{\partial \rho_1}{\partial t} = -\rho_0 u_1, \quad (14)$$

where we have restricted ourselves to the 1-D case with  $\mathbf{u} = \mathbf{u}(x) = (u(x), 0, 0)$ . We write the pressure gradient in the form

$$\frac{\partial P_1}{\partial x} = \frac{P_0}{T_0} \frac{\partial T_1}{\partial x} + \frac{P_0}{\rho_0} \frac{\partial \rho_1}{\partial x}, \quad (15)$$

so the linearized momentum equation,

$$\rho_0 \frac{\partial u_1}{\partial t} = -\frac{\partial P_1}{\partial x} = -\frac{P_0}{T_0} \frac{\partial T_1}{\partial x} - \frac{P_0}{\rho_0} \frac{\partial \rho_1}{\partial x}, \quad (16)$$

It is convenient to introduce the (adiabatic) sound speed  $c_s$  via  $c_s^2 = \gamma P/\rho$ . The equilibrium value is referred to as  $c_{s0}$ . Thus, we have

$$\frac{\partial u_1}{\partial t} = -\frac{c_{s0}^2}{T_0} \frac{\partial T_1}{\partial x} - \frac{c_{s0}^2}{\rho_0} \frac{\partial \rho_1}{\partial x}, \quad (17)$$

The linearized temperature equation can be written as

$$c_v \rho_0 \frac{\partial T_1}{\partial t} - c_p T_0 \left(1 - \frac{1}{\gamma}\right) \frac{\partial \rho_1}{\partial t} = -\rho_0 \mathcal{L}_\rho \rho_1 - \rho_0 \mathcal{L}_T T_1 + K \nabla^2 T_1. \quad (18)$$

Dividing by  $\rho_0 c_v$  and  $T_0$ , we have

$$\frac{1}{T_0} \frac{\partial T_1}{\partial t} - (\gamma - 1) \frac{1}{\rho_0} \frac{\partial \rho_1}{\partial t} = -\frac{\mathcal{L}_\rho}{c_v T_0} \rho_1 - \frac{\mathcal{L}_T}{c_v} T_1/T_0 + \gamma \chi \nabla^2 T_1/T_0. \quad (19)$$

where  $\chi = K/(\rho c_p)$  is the radiative *diffusivity* (as opposed to the radiative conductivity).<sup>1</sup>

Collecting the three equations and making the usual ansatz  $\rho_1 = \hat{\rho}_1 e^{\sigma t + i\mathbf{k} \cdot \mathbf{x}}$ ,  $u_1 = \hat{u}_1 e^{\sigma t + i\mathbf{k} \cdot \mathbf{x}}$ ,  $T_1 = \hat{T}_1 e^{\sigma t + i\mathbf{k} \cdot \mathbf{x}}$ , leads to the following *system of algebraic equations*:

$$\sigma \rho_1 / \rho_0 + i k u_1 = 0, \quad (20)$$

$$\sigma u_1 + i k c_{s0}^2 (\rho_1 / \rho_0 + T_1 / T_0) = 0, \quad (21)$$

$$\sigma T_1 / T_0 - \sigma (\gamma - 1) \rho_1 + \frac{\mathcal{L}_\rho}{c_v T_0} \rho_1 + \frac{\mathcal{L}_T}{c_v} \frac{T_1}{T_0} = 0. \quad (22)$$

It is useful to define  $\sigma_\rho = \rho_0 \mathcal{L}_\rho / c_v T_0$  and  $\sigma_T = \mathcal{L}_T / c_v$ , so that we have

$$(\sigma + \sigma_T) T_1 / T_0 - [\sigma (\gamma - 1) - \sigma_\rho] \rho_1 / \rho_0 = 0. \quad (23)$$

It is convenient to rewrite this in matrix form,  $\mathbf{M} \mathbf{q} = 0$ , for state vector  $\mathbf{q} = (\rho_1 / \rho_0, u_1, s_1 / T_0)^T$ ,

$$\begin{pmatrix} \sigma & i k & 0 \\ i k c_{s0}^2 & \sigma & i k c_{s0}^2 \\ \sigma_\rho - \sigma (\gamma - 1) & 0 & \sigma + \sigma_T \end{pmatrix} \begin{pmatrix} \rho_1 / \rho_0 \\ u_1 \\ T_1 / T_0 \end{pmatrix} = 0. \quad (24)$$

which leads to the dispersion relation

$$\sigma^2 (\sigma + \sigma_T) + k^2 c_{s0}^2 [\sigma (\gamma - 1) - \sigma_\rho] + k^2 c_{s0}^2 (\sigma + \sigma_T) = 0. \quad (25)$$

which simplifies to

$$\sigma^2 (\sigma + \sigma_T) + k^2 c_{s0}^2 (\sigma \gamma - \sigma_\rho + \sigma_T) = 0. \quad (26)$$

## 5 Power law cooling function

It is convenient to work with piecewise power laws, i.e.,

$$\mathcal{L} = \rho \Lambda(T) - \Gamma, \quad \Lambda(T) = \Lambda_0 T^\beta. \quad (27)$$

Let us also assume that  $\Gamma = \text{const}$ , so that

$$\mathcal{L}_\rho = \Lambda_0 \quad \text{and} \quad \mathcal{L}_T = \rho \beta \Lambda_0 / T. \quad (28)$$

Since  $\sigma_\rho = \rho_0 \mathcal{L}_\rho / c_v T_0$  and  $\sigma_T = \mathcal{L}_T / c_v$ , we have

$$\sigma_\rho = \frac{\rho_0 \Lambda_0}{c_v T_0}, \quad \sigma_T = \beta \frac{\rho_0 \Lambda_0}{c_v T_0} = \beta \sigma_\rho. \quad (29)$$

Thus, the dispersion relation becomes

$$\sigma^2 (\sigma + \beta \sigma_\rho) + c_s^2 k^2 [\gamma \sigma + (\beta - 1) \sigma_\rho] = 0 \quad (30)$$

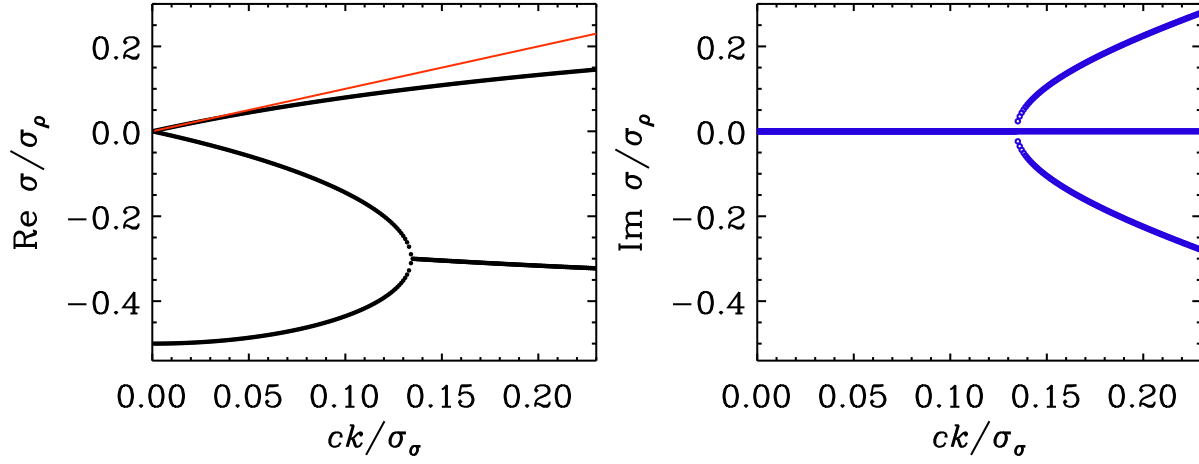


Figure 1: All three roots of Equation (30).

An approximate solution can be obtained by writing

$$\sigma^2 \approx -c_s^2 k^2 \frac{\gamma\sigma + (\beta - 1)\sigma_\rho}{\sigma + \beta\sigma_\rho} \rightarrow \sigma^2 \approx c_s^2 k^2 \frac{1 - \beta}{\beta} \quad (\text{for } \sigma \ll \sigma_\rho) \quad (31)$$

In Figure 1 we compare this approximation with the real and imaginary parts of all three roots.

We see that the instability criterion is  $\beta < 1$ . This covers a broader range of cases than the isochoric stability criterion, which corresponds to  $\beta < 0$ . The following section illustrates the instability for a parameterized (and yet realistic) cooling function. Full details can be found in Brandenburg et al. (2007).

<sup>1</sup>We'll return later to the question why it is defined with  $c_p$  rather than  $c_v$ .

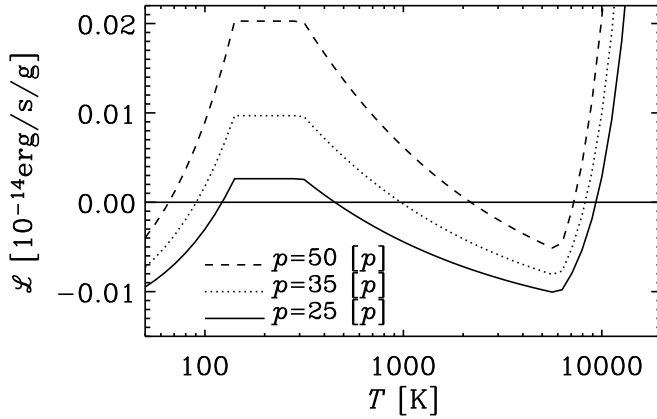


Figure 2: Net cooling vs. temperature for three values of  $p$ , given in units of  $[p] = 10^{-14}$  dyn.

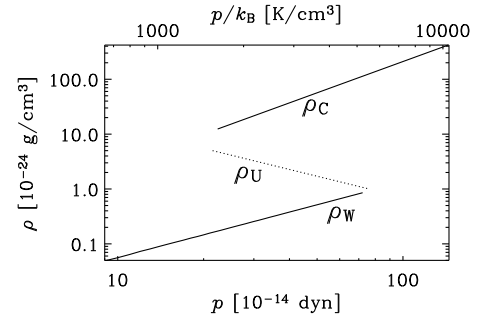


Figure 3: The two stable solution branches,  $\rho_C$  and  $\rho_W$  (solid lines), and the unstable solution branch,  $\rho_U$  (dotted line), as a function of  $p$ . On the top the pressure is normalized by the Boltzmann constant,  $p/k_B$ .

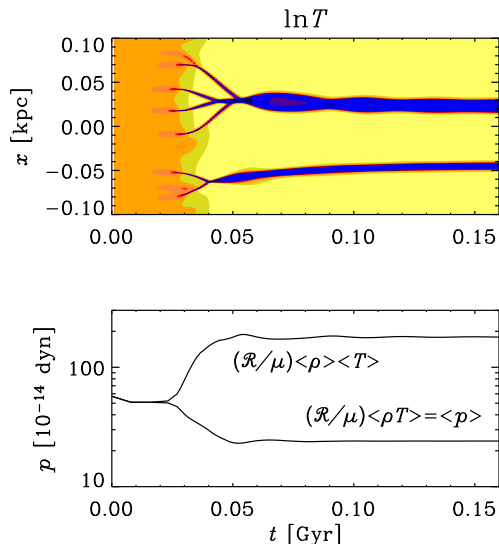


Figure 4: Evolution of  $\ln T$  in a space-time diagram (top) and of the mean pressure (bottom) in a one-dimensional simulation with 1024 mesh points and  $\nu = \chi = 5 \times 10^{-4} \text{ Gyr } k_m^2 \text{ s}^{-2}$ . During early times, the rms velocity grows exponentially at a rate of about  $220 \text{ Gyr}^{-1}$ .

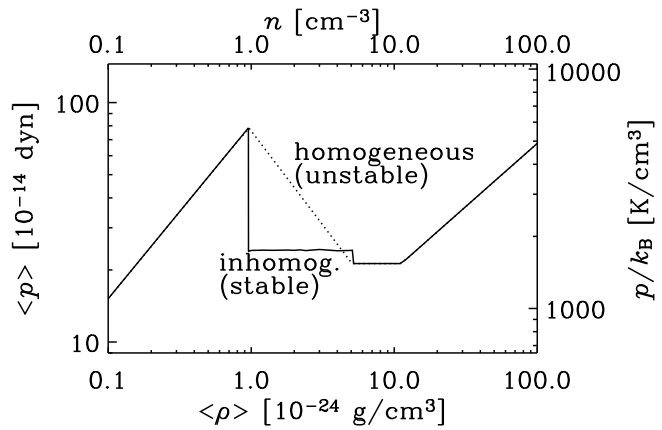


Figure 5: Mean pressure vs. mean density in a one-dimensional simulation (solid line), compared with the values obtained for a homogeneous system (dotted line). The dotted line (which agrees with the solid line in the stable regime and hence cannot be seen there) was obtained by plotting  $p(T)$  vs.  $\rho(T)$  using  $T$  as a parameter in Equation (33). On the right axis, the pressure is normalized by the Boltzmann constant,  $p/k_B$ , and at the top the number density is given. The simulation has 128 mesh points and  $\nu = \chi = 5 \times 10^{-3} \text{ Gyr } k_m^2 \text{ s}^{-2}$ .

## 6 Piecewise power law cooling function

To illustrate properties of the thermal instability in the interstellar medium, let us adopt a parameterization of the cooling function approximately equal to that given by Sánchez-Salcedo et al. (2002), which has been obtained by fitting a piecewise power law function of the form

$$\Lambda(T) = C_{i,i+1} T^{\beta_{i,i+1}} \quad \text{for } T_i \leq T < T_{i+1}, \quad (32)$$

In Figure 2 we plot  $\mathcal{L}$  as a function of  $T$  for constant  $p$ ; three values are considered:  $p = 25, 35,$  and  $50$ , all in units of  $[p] \equiv 10^{-14} \text{ dyn}$ . This figure shows that there can be two stable states at about  $10^2$  and  $10^4 \text{ K}$ . We denote these values by  $T_C$  and  $T_W$  for the cold and warm phases. At  $T \approx 10^3 \text{ K}$  there is an unstable equilibrium, whose temperature is denoted  $T_U$ . The densities of the three equilibria, obtained by solving  $\mathcal{L}(T; P) = 0$  for  $T$  numerically for given  $P$  and then expressing the result in terms of  $\rho = \rho(T, P)$ , are plotted in Figure 3.

When the mean density is outside the range between 0.96 and 5.1 (in units of  $10^{-24} \text{ g cm}^{-3}$ ), the gas is thermally stable and remains uniform. The dependence of the pressure on the density can be obtained in parametric form by calculating, using temperature as a parameter,  $\rho(T)$  and  $p(T)$ , that is,

$$\rho(T) = \frac{\Gamma}{\Lambda(T)}, \quad p(T) = \frac{\mathcal{R}T}{\mu} \frac{\Gamma}{\Lambda(T)}, \quad (33)$$

and plotting the two against each other (see Figure 5, dotted line). The numerically obtained values for the mean pressure  $\langle p \rangle$ , for different mean densities  $\langle \rho \rangle$ , agree with those obtained under the assumption of homogeneity.

In Figure 4 we plot the evolution of  $\ln T$  in a space-time diagram (top) and that of the mean pressure in a one-dimensional simulation. Here,  $\nu = \chi = 5 \times 10^{-4} \text{ Gyr } k_m^2 \text{ s}^{-2}$  which, together with the initial values of  $c_s = 7.5 \text{ km/s}$  and  $n_\rho = 980 \text{ Gyr}^{-1}$ , yields  $k_F = 720 \text{ kpc}^{-1} = 23k_1$ , and hence  $n_\rho / (c_s k_F) \approx 0.2$ .

In the unstable regime the pressure is, surprisingly, independent of  $\langle\rho\rangle$ , and always around  $\langle p\rangle \approx 24.2 \times 10^{-14}$  dyn. Figure 3 shows that for  $\langle p\rangle \approx 24.2 \times 10^{-14}$  dyn the warm and cool phases have  $\rho_W \approx 0.19$  and  $\rho_C \approx 14.3$ , respectively.

## References

- Brandenburg, A., Korpi, M. J., & Mee, A. J., “Thermal instability in shearing and periodic turbulence,” *Astrophys. J.* **654**, 945-954 (2007).
- Field, G. B., “Thermal instability,” *Astrophys. J.* **142**, 531-567 (1965).
- Sánchez-Salcedo, F. J., Vázquez-Semadeni, E., & Gazol, A., “The nonlinear development of the thermal instability in the atomic interstellar medium and its interaction with random fluctuations,” *Astrophys. J.* **577**, 768-788 (2002).
- E. N. Parker, “Instability of thermal fields,” *Astrophys. J.* **117**, 431-436 (1953).