Handout 20: Turbulent magnetic diffusivity &

In Homework 4, we showed that

$$\mathcal{E}_i = \alpha_{ip}\overline{B}_p + \eta_{ipl}\overline{B}_{p,l}. \tag{1}$$

We then looked at the *isotropic* reduction, $\alpha = \frac{1}{3}\delta_{ip}\alpha_{ip}$, and found that

$$\alpha = -\tfrac{1}{3}\tau\overline{\boldsymbol{\omega}\cdot\boldsymbol{u}}. \tag{2}$$

where $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\boldsymbol{u}$ is the vorticity. Let us now compute the η_{ipl} tensor.

1 Analytic calculation of the η_{ipl} tensor

In Homework 4, we used

$$\mathcal{E}_i = \epsilon_{ijk}\epsilon_{klm}\epsilon_{mnp}\,\tau\overline{u_j\partial_l u_n\overline{B}_p} \tag{3}$$

to derive Equation (1), which defines rank 2 and rank 3 tensors. We found that $\alpha_{ip} = \epsilon_{jnp}\,\tau\overline{u_j u_{n,i}} - \epsilon_{inp}\,\tau\overline{u_j u_{n,j}}$. Let us now also consider the η_{ipl} tensor, which is given by

$$\eta_{ipl} = \epsilon_{ijk}\epsilon_{klm}\epsilon_{mnp}\,\tau\overline{u_j u_n}. \tag{4}$$

Using $\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$, we have

$$\eta_{ipl} = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\epsilon_{mnp}\,\tau\overline{u_j u_n}, \tag{5}$$

so

$$\eta_{ipl} = \delta_{il}\epsilon_{jnp}\,\tau\overline{u_j u_n} - \epsilon_{inp}\,\tau\overline{u_l u_n}. \tag{6}$$

The first term drops out, because $\epsilon_{jnp}\,\tau\overline{u_j u_n} = 0$. We are then left with

$$\eta_{ipl} = -\epsilon_{inp}\,\tau\overline{u_l u_n}. \tag{7}$$

To make some sense of this, let us consider the isotropic part. As discused last time, we assume $\eta_{ipl} = \eta_{\rm t}\epsilon_{ipl}$ and determine $\eta_{\rm t} = \frac{1}{6}\epsilon_{ipl}\eta_{ipl}$, i.e.,

$$\eta_{\rm t} = -\tfrac{1}{6}\epsilon_{ipl}\epsilon_{inp}\,\tau\overline{u_l u_n}. \tag{8}$$

Let's rearrange the indices on ϵ_{ipl} in circular order, so

$$\eta_{\rm t} = -\tfrac{1}{6}\epsilon_{pli}\epsilon_{inp}\,\tau\overline{u_l u_n}, \tag{9}$$

and thus

$$\eta_{\rm t} = -\tfrac{1}{6}\underbrace{(\delta_{pn}\delta_{lp} - \delta_{pp}\delta_{ln})}_{-2\delta_{ln}}\,\tau\overline{u_l u_n} = \tfrac{1}{3}\tau\overline{\boldsymbol{u}^2} \tag{10}$$

Table 1: Typical numbers for the solar convection zone.

$u_{\rm rms}$ [m/s]	H_P [m]	$\eta_{\rm t}$ [$\mathrm{m^2\,s^{-1}}$]	$\eta_{\rm t}$ [$\mathrm{cm^2\,s^{-1}}$]
1000	300×10^3	1×10^8	1×10^{12}
20	50×10^6	3×10^8	3×10^{12}

Typical values of $\eta_{\rm t}$ greatly exceed η if the magnetic Reynolds number Re_M is large. In fact, if we also estimate $\tau = (u_{\rm rms}k_{\rm f})^{-1}$, then $\mathrm{Re}_M/3 = \eta_{\rm t}/\eta$. In the Sun, we can identify $1/k_{\rm f}$ with the mixing length, which increases approximately linearly with depth from 300 km near the surface to 50 Mm near the bottom of the convection zone.

2 Magnetic quenching

If the magnetic field becomes strong (magnetic energy density comparable to the kinetic energy density), the value of $\eta_{\rm t}$ becomes *quenched*. How strong this effect is has been a matter of debate for decades (Piddington, 1972; Knobloch, 1978; Piddington, 1981; Cattaneo & Vainshtein, 1991). Most of this work is analytic, and only the work of Cattaneo & Vainshtein (1991) is numerical, but it is restricted to 2-D, where the conservation of $\langle A^2 \rangle$ strongly affects the result, as was understood later (Gruzinov & Diamond, 1994). A relatively new method for computing turbulent transport coefficients such as α_{ij} and η_{ijk} is the test-field method. Figure 1 shows the dependence of $\eta_{\rm t}$ on Re_M (Brandenburg et al., 2008), where $\tilde{\alpha} = \alpha/\alpha_0$ and $\tilde{k}_{\rm f} = k_{\rm f}/k_1$ with $\alpha_0 = -\frac{1}{3} u_{\rm rms}(\overline{\boldsymbol{B}})$ have been used for normalization. In this simulations the field is sustained against turbulent decay by a self-consistent α effect such that $\tilde{\lambda} \equiv \lambda/(\eta_{t0} k_1^2) = \tilde{\alpha}\tilde{k}_{\rm f} - (\tilde{\eta}_{\rm t} + \tilde{\eta})$ is approximately zero.

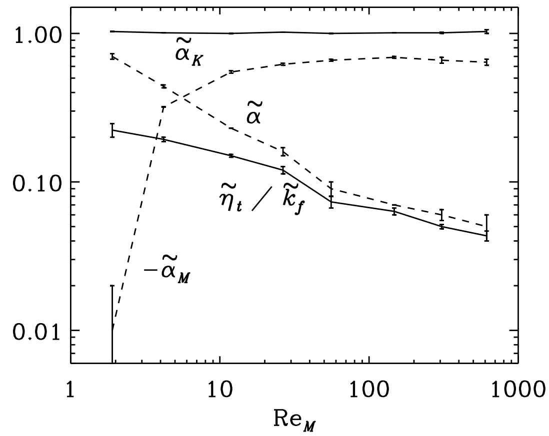

Figure 1: Re_M-dependence of $\tilde{\alpha}$ and $\tilde{\eta}_{\rm t}/\tilde{k}_{\rm f}$.

The structure of the turbulence is determined by the vectors $\overline{\boldsymbol{B}}$ and $\overline{\boldsymbol{J}}$, but for a Beltrami field they are aligned, so we have

$$\alpha_{ij}(\overline{\boldsymbol{B}}) = \alpha_1(\overline{\boldsymbol{B}})\delta_{ij} + \alpha_2(\overline{\boldsymbol{B}})\hat{\overline{B}}_i\hat{\overline{B}}_j, \tag{11}$$

$$\eta_{ij}(\overline{\boldsymbol{B}}) = \eta_1(\overline{\boldsymbol{B}})\delta_{ij} + \eta_2(\overline{\boldsymbol{B}})\hat{\overline{B}}_i\hat{\overline{B}}_j, \tag{12}$$

where $\hat{\boldsymbol{B}}$ means the unit vector in the direction of $\overline{\boldsymbol{B}}$.

The lifetime of sunspots is between a day (for small spots) to 3 months. The decay of sunspots can be modelled as a turbulent decay (Krause & Rüdiger, 1975; Petrovay & van Driel-Gesztelyi, 1997; Rüdiger & Kitchatinov, 2000). Estimating the sunspot decay as $T_{\rm decay} = (\eta_{\rm t} k_1^2)^{-1}$ and using $\eta_{\rm t} = 10^8\,{\rm m^2\,s^{-1}}$ and $1/k_1 = 30\,{\rm Mm}$, we find 100 days. A 3 times smaller spot would decay 10 times faster, so these estimates are rather sensitive to size.

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