Handout 22: Turbulent transport

By turbulent transport one usually means the effect of turbulence on the averaged properties of the system. The turbulent transport of a passive scalar is in many ways the simplest problem, but in some ways it is not sufficiently generic to be able to identify the broader structure behind this concept. A better example that we have already encountered is the turbulent transport of magnetic field. (We have also already encountered the turbulent transport of momentum, but that is a harder problem if one wants to do it rigorously.

A general way of computing α and $\eta_{\rm t}$ is the test-field method. It was invented by Schrinner et al. (2005, 2007) and since then be used for many different applications (Brandenburg, 2005; Brandenburg et al., 2008).

1 Test-field method

The theory of turbulent resistivity is in many ways more developed than the theory of turbulent viscosity. In dynamo theory it has recently become possible to determine quite accurately not only the turbulent resistivity, but also its full tensorial form and other components that can be non-dissipative and hence important for dynamo action. While in the hydrodynamic case one is interested in the correlation $\overline{u_i u_j}$, one is here interested in the correlation $\overline{u_j b_j}$, or more specifically in the electromotive force $\overline{\mathcal{E}}_i = \epsilon_{ijk}\overline{u_j b_j}$. Assuming that the mean field is spatially smooth (which may not be the case in practice) one can truncate the expression for $\overline{\mathcal{E}}_i$ in terms of $\overline{B}_j$ and its derivatives after the first derivative, so one has

$$\overline{\mathcal{E}}_i = \alpha_{ij}\overline{B}_j + \eta_{ijk}\overline{B}_{j,k}. \tag{1}$$

The components of α_{ij} tensor are usually quite easily determined from simulations by imposing a uniform magnetic field $\overline{B}_j$ and measuring the resulting electromotive force $\overline{\mathcal{E}}_i$, so that $\alpha_{ij} = \overline{\mathcal{E}}_i/\overline{B}_j$ is obtained straightforwardly. The reason this works is because for a uniform field all derivatives of $\overline{B}_j$ vanish, so there are no higher order terms. Calculating the components of η_{ijk} is usually harder, especially when the mean field may no longer be smooth and its derivatives may vanish in places. A method that has been used for accretion disc turbulence is based on a fitting procedure of the measured mean field and the mean electromotive force to Equation (1) by calculating moments of the form $\langle\overline{\mathcal{E}}_i\overline{B}_j\rangle$, $\langle\overline{\mathcal{E}}_i\overline{B}_{j,k}\rangle$, as well as $\langle\overline{B}_i\overline{B}_j\rangle$ and $\langle\overline{B}_i\overline{B}_{j,k}\rangle$.

A general procedure for determining the full α_{ij} and η_{ijk} tensors from a simulation is to calculate the electromotive force after applying test fields of different directions and with different gradients (Schrinner et al. 2005). In the following we adopt xy averages, so the resulting mean fields depend only on z and t, and only $\overline{B}_x$ and $\overline{B}_y$ are non-trivial ($\overline{B}_z = 0$ because of the solenoidality $\overline{\boldsymbol{B}}$). Therefore, only the four components of α_{ij} and the four components of η_{ij3} with $i, j = 1, 2$ are non-trivial. Here, the numbers $1, 2, 3$ refer to the cartesian x, y, z components.

In the present case of one-dimensional mean fields it is advantageous to rewrite Equation (1) in the form

$$\overline{\mathcal{E}}_i = \alpha_{ij}\overline{B}_j - \tilde{\eta}_{ij}\overline{J}_j, \quad i, j = 1, 2, \tag{2}$$

where $\overline{\boldsymbol{J}} = \boldsymbol{\nabla}\times\overline{\boldsymbol{B}}$ is the mean current density, and

$$\tilde{\eta}_{il} = \eta_{ijk}\epsilon_{jkl} \tag{3}$$

is the resistivity tensor operating only on the mean current density. In the special case of one-dimensional averages there is no extra information contained in the symmetric part of the $\overline{B}_{j,k}$ tensor that is not already contained in the components of $\overline{\boldsymbol{J}}$. In fact, the four components of η_{ij3} map uniquely to those of $\tilde{\eta}_{il}$ via

$$\begin{pmatrix} \tilde{\eta}_{11} & \tilde{\eta}_{12} \\ \tilde{\eta}_{21} & \tilde{\eta}_{22} \end{pmatrix} = \begin{pmatrix} \eta_{123} & -\eta_{113} \\ \eta_{223} & -\eta_{213} \end{pmatrix}. \tag{4}$$

This fact was also used in Brandenburg & Sokoloff (2002). The diagonal components of $\tilde{\eta}_{ij}$ correspond to turbulent resistivity, while its off-diagonal components can be responsible for driving dynamo action [$\boldsymbol{\Omega}\times\overline{\boldsymbol{J}}$ and $\overline{\boldsymbol{W}}\times\overline{\boldsymbol{J}}$ effects; see, Rädler (1969) and Rogachevskii & Kleeorin (2003, 2004), Rädler &

Stepanov (2005)]. Conversely, the diagonal components of the α tensor can be responsible for dynamo action while the off-diagonal components are responsible for non-regenerative turbulent pumping effects (Krause & Rädler 1980). It should be noted, however, that for linear shear flows Rüdiger & Kitchatinov (2005) find that the signs of the relevant coefficients of $\tilde{\eta}_{ij}$ are such that dynamo action is not possible for small magnetic Prandtl numbers.

In summary, in the present case of one-dimensional mean fields, $\overline{\boldsymbol{B}} = \overline{\boldsymbol{B}}(z,t)$, there are altogether $4+4$ unknowns. The idea is to calculate the electromotive force

$$\overline{\boldsymbol{\mathcal{E}}}^{(p,q)} = \overline{\boldsymbol{u} \times \boldsymbol{b}^{(p,q)}} \tag{5}$$

for the excess magnetic fluctuations, $\boldsymbol{b}^{(p,q)}$, that are due to a given test field $\overline{\boldsymbol{B}}^{(p,q)}$, where the labels p and q characterize the test field (p gives its nonvanishing component and $q=1$ or 2 stands for cosine or sine-like test fields. The calculation of the electromotive force requires solving simultaneously a set of equations of the form

$$\frac{\partial \boldsymbol{b}^{(p,q)}}{\partial t} = \boldsymbol{\nabla} \times \left[\left(\overline{\boldsymbol{U}} \times \boldsymbol{b}^{(p,q)} + \boldsymbol{u} \times \overline{\boldsymbol{B}}^{(p,q)} \right) \right] + \eta \nabla^2 \boldsymbol{b}^{(p,q)} + \boldsymbol{G} \tag{6}$$

for each test field $\overline{\boldsymbol{B}}^{(p,q)}$. Here, $\boldsymbol{G} = \boldsymbol{\nabla} \times [\boldsymbol{u} \times \boldsymbol{b}^{(p,q)} - \overline{\boldsymbol{u} \times \boldsymbol{b}^{(p,q)}}]$ is a term nonlinear in the fluctuation. This term would be ignored in the first order smoothing approximation, but it can be kept in a simulation if desired. (In the present considerations it is neglected.)

The four test fields considered in the present problem of one-dimensional mean fields are

$$\overline{\boldsymbol{B}}^{(1,1)} = \begin{pmatrix} \cos k_1 z \\ 0 \\ 0 \end{pmatrix}, \quad \overline{\boldsymbol{B}}^{(1,2)} = \begin{pmatrix} \sin k_1 z \\ 0 \\ 0 \end{pmatrix}, \tag{7}$$

$$\overline{\boldsymbol{B}}^{(2,1)} = \begin{pmatrix} 0 \\ \cos k_1 z \\ 0 \end{pmatrix}, \quad \overline{\boldsymbol{B}}^{(2,2)} = \begin{pmatrix} 0 \\ \sin k_1 z \\ 0 \end{pmatrix}. \tag{8}$$

As an example, consider the x component of $\overline{\boldsymbol{\mathcal{E}}}^{(p,q)}$ for $p=1$ and both values of q,

$$\overline{\mathcal{E}}_1^{(1,1)} = \alpha_{11} \cos k_1 z - \eta_{113} \sin k_1 z, \tag{9}$$

$$\overline{\mathcal{E}}_1^{(1,2)} = \alpha_{11} \sin k_1 z + \eta_{113} \cos k_1 z. \tag{10}$$

For $p=2$, and/or for $i=2$, one obtains a similar pair of equations with the same arrangement of cosine and sine functions. So, for each of the four combinations of i and j $(=p)$ the set of two coefficient, α_{ij} and η_{ij3}, is obtained as

$$\begin{pmatrix} \alpha_{ij} \\ \eta_{ij3} \end{pmatrix} = \mathsf{M}^{-1} \begin{pmatrix} \overline{\mathcal{E}}_i^{(j,1)} \\ \overline{\mathcal{E}}_i^{(j,2)} \end{pmatrix}, \tag{11}$$

where the matrix

$$\mathsf{M} = \begin{pmatrix} \cos k_1 z & -\sin k_1 z \\ \sin k_1 z & \cos k_1 z \end{pmatrix} \tag{12}$$

is the same for each value of p and each of the two components $i=1,2$ of $\overline{\mathcal{E}}_i^{(p,q)}$. Finally, $\tilde{\eta}$ is calculated using Equation (3). Note that $\det\mathsf{M} = 1$, so the inversion procedure is well behaved and even trivial.

References

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