Handout 2: Magnetorotational Instability

The magnetorotational instability (MRI) is another relatively recent addition to our repertoire of known instabilities. It's discovery goes back to Velikhov (1959), but the astrophysical significance was discovered only more recently by Balbus & Hawley (1991).

This instability is also interesting in the sense that there is a natural transition between waves and turbulence as one changes some relevant control parameter. To understand all this, let us focus on the essentials of this instability. We do this by neglecting gas pressure. As we will see below, in the case of a vertical field, the pressure just enters in connection with sound waves, which leads to a completely separate branch of the final dispersion relation. Owing to this decoupling, the results derived below turn out to be exactly the same as those obtained when the pressure is retained.

1 Background shear flow

Before writing down the governing equations, we need to discuss the geometry under consideration. We want to develop a local model of a rotating shear (or Couette) flow. Shear is here the result of the balance between gravity inward, i.e., $-GM/R$ and centrifugal force outward, $\boldsymbol{\Omega}^2 R$, where $\boldsymbol{\Omega}$ is the angular velocity and R is the distance from the axis. In addition, since we are in a rotating system, whose angular velocity is $\boldsymbol{\Omega}$, we have to consider the Coriolis force $2\boldsymbol{\Omega}\times\boldsymbol{u}$. To understand this basic balance, let us write down the equations in the form

$$0 = -2\boldsymbol{\Omega}\times\boldsymbol{U} - \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{R}) - \frac{GM}{R^3}\boldsymbol{R}.$$

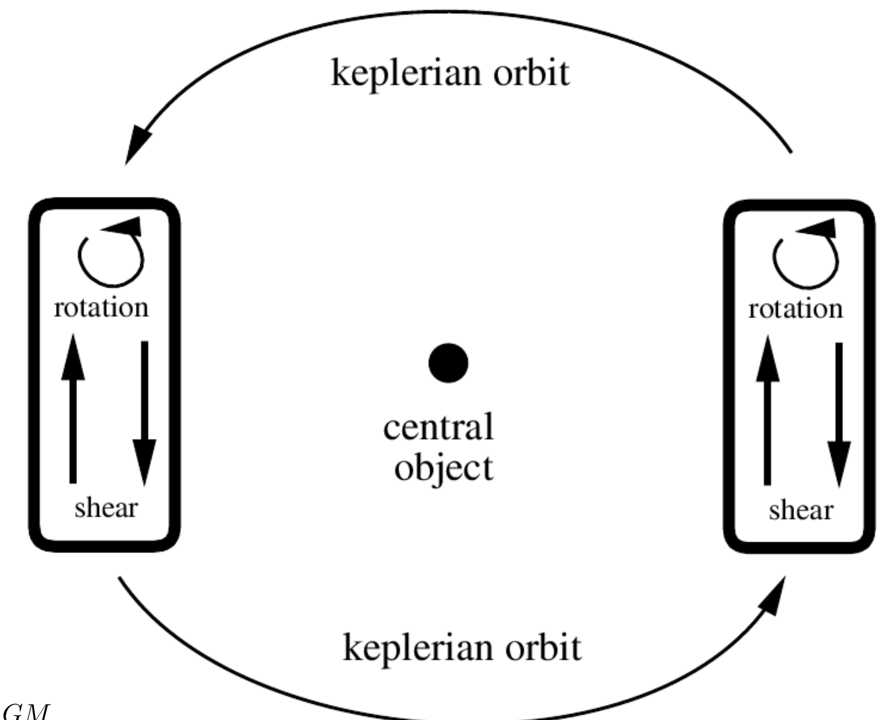

Figure 1: Shearing sheet geometry.

or, in component form,

$$0 = -\begin{pmatrix}0\\0\\2\Omega\end{pmatrix}\times\begin{pmatrix}U_x\\U_y\\U_z\end{pmatrix} - \begin{pmatrix}0\\0\\\Omega\end{pmatrix}\times\left[\begin{pmatrix}0\\0\\\Omega\end{pmatrix}\times\begin{pmatrix}R\\0\\0\end{pmatrix}\right] - \frac{GM}{R^3}\begin{pmatrix}R\\0\\0\end{pmatrix}. \tag{1}$$

This gives

$$0 = \begin{pmatrix}+2\Omega U_y\\-2\Omega U_x\\0\end{pmatrix} - \begin{pmatrix}0\\0\\\Omega\end{pmatrix}\times\begin{pmatrix}0\\\Omega R\\0\end{pmatrix} - \begin{pmatrix}GM/R^2\\0\\0\end{pmatrix}. \tag{2}$$

and thus

$$0 = \begin{pmatrix}+2\Omega U_y\\-2\Omega U_x\\0\end{pmatrix} + \begin{pmatrix}\Omega^2 R\\0\\0\end{pmatrix} - \begin{pmatrix}GM/R^2\\0\\0\end{pmatrix}. \tag{3}$$

So obviously $U_x = 0$. Furthermore, let us consider a local coordinate relative to R by setting $R = R_0 + x$, where $x \ll R_0$. Expanding the fraction yields the following for the x component of this vector

$$0 = 2\Omega U_y + \Omega^2 R(1 + x/R) - (GM/R^2)(1 - 2x/R). \tag{4}$$

This leads to $\Omega^2 R = GM/R^2$, or $\Omega^2 = GM/R^3$, which is known as Kepler's law (here for the simple case of a circular orbit). The rest of this equation then becomes

$$0 = 2\Omega U_y + \Omega^2 x + 2GM/R^3 x, \tag{5}$$

or, because of $GM/R^3 = \Omega^2$,

$$0 = 2\Omega U_y + 3\Omega^2 x. \tag{6}$$

This yields

$$U_y = U_y(x) = -\tfrac{3}{2}\Omega x \equiv Sx, \tag{7}$$

where we have introduced the shear rate $S = -\frac{3}{2}\Omega$. So this is our shear flow. Note that it is by construction a linear shear flow.

Next, returning to the full flow speed, $\boldsymbol{U} + \boldsymbol{u}$, and inserting it into the total time derivative, we have

$$\text{time derivative} = \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{U} + \boldsymbol{u}) \cdot \boldsymbol{\nabla}(\boldsymbol{U} + \boldsymbol{u}), \tag{8}$$

where we have used the fact that the shear flow does not depend on t. Thus, we have

$$\text{time derivative} = \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{U} \cdot \boldsymbol{\nabla}\boldsymbol{U} + \boldsymbol{U} \cdot \boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{u} \cdot \boldsymbol{\nabla}\boldsymbol{U} + \boldsymbol{u} \cdot \boldsymbol{\nabla}\boldsymbol{u} = \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{U} \cdot \boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{u} \cdot \boldsymbol{\nabla}\boldsymbol{U} + \boldsymbol{u} \cdot \boldsymbol{\nabla}\boldsymbol{u}, \tag{9}$$

where we have used the fact that $\boldsymbol{U} \cdot \boldsymbol{\nabla}\boldsymbol{U} = 0$, because $\boldsymbol{U}$ has only a y component, but $\boldsymbol{U}$ does not depend on y, so $\partial\boldsymbol{U}/\partial y = 0$. This yields

$$\text{time derivative} = \frac{\partial \boldsymbol{u}}{\partial t} + Sx\frac{\partial}{\partial y}\boldsymbol{u} + u_x S\hat{\boldsymbol{y}} + \boldsymbol{u} \cdot \boldsymbol{\nabla}\boldsymbol{u}. \tag{10}$$

Thus, the presence of shear leads to 2 new terms; one being an advection term in the y direction and the other one a stretching term. This advection term will appear in all equations (e.g., in the continuity equation and the induction equation), but the stretching term will only appear in the induction equation. With these preparations we can now write down the full set of governing equations.

2 Governing equations

In the presence of rotation and shear, the MHD equations for the departure from the shear flow $\boldsymbol{u}$, the magnetic field $\boldsymbol{B}$, and the density ρ becomes

$$\frac{\partial \boldsymbol{u}}{\partial t} + Sx\frac{\partial \boldsymbol{u}}{\partial y} + u_x S\hat{\boldsymbol{y}} + \boldsymbol{u} \cdot \boldsymbol{\nabla}\boldsymbol{u} + 2\boldsymbol{\Omega} \times \boldsymbol{u} = -\rho^{-1}\boldsymbol{\nabla}P + \rho^{-1}\boldsymbol{J} \times \boldsymbol{B}, \tag{11}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + Sx\frac{\partial \boldsymbol{B}}{\partial y} + \boldsymbol{u} \cdot \boldsymbol{\nabla}\boldsymbol{B} = B_x S\hat{\boldsymbol{y}} + \boldsymbol{B} \cdot \boldsymbol{\nabla}\boldsymbol{u} - \boldsymbol{B}\boldsymbol{\nabla} \cdot \boldsymbol{u}, \tag{12}$$

$$\frac{\partial \rho}{\partial t} + Sx\frac{\partial \rho}{\partial y} + \boldsymbol{u} \cdot \boldsymbol{\nabla}\rho = -\rho\boldsymbol{\nabla} \cdot \boldsymbol{u}. \tag{13}$$

Let us here, for simplicity, consider an isothermal equation of state, i.e., $P = \rho c_{\rm s}^2$, where $c_{\rm s} = \text{const}$. The equations can be readily linearized about $\boldsymbol{u} = 0$, $\boldsymbol{B} = \boldsymbol{B}_0 = \text{const}$, and $\rho = \rho_0 = \text{const}$. For the following, we assume $\boldsymbol{B}_0 = (0, 0, B_0)$ and $\boldsymbol{\nabla} = (0, 0, \partial_z)$. If we were to allow for y derivatives, our equations have non-constant coefficients, which complicates the analysis. We assume that all perturbations are proportional to $e^{\sigma t+\mathrm{i}kz}$. Thus we have

$$\begin{pmatrix} \sigma & -2\Omega & 0 & 0 & -\mathrm{i}k\frac{B_0}{\mu_0\rho_0} & 0 \\ S+2\Omega & \sigma & 0 & 0 & 0 & -\mathrm{i}k\frac{B_0}{\mu_0\rho_0} \\ & 0 & \sigma & \mathrm{i}kc_\mathrm{s}^2 & 0 & 0 \\ 0 & 0 & \mathrm{i}k & \sigma & 0 & 0 \\ -\mathrm{i}kB_0 & 0 & 0 & 0 & \sigma & 0 \\ 0 & -\mathrm{i}kB_0 & 0 & 0 & -S & \sigma \end{pmatrix} \begin{pmatrix} \hat{u}_{x1} \\ \hat{u}_{y1} \\ \hat{u}_{z1} \\ \hat{\rho}_1/\rho_0 \\ \hat{B}_{x1} \\ \hat{B}_{y1} \end{pmatrix} = 0. \tag{14}$$

We see now that sound waves decouple, so we can simplify the matrix to

$$\begin{pmatrix} \sigma & -2\Omega & -\mathrm{i}k\frac{B_0}{\mu_0\rho_0} & 0 \\ S+2\Omega & \sigma & 0 & -\mathrm{i}k\frac{B_0}{\mu_0\rho_0} \\ -\mathrm{i}kB_0 & 0 & \sigma & 0 \\ 0 & -\mathrm{i}kB_0 & -S & \sigma \end{pmatrix} \begin{pmatrix} \hat{u}_{x1} \\ \hat{u}_{y1} \\ \hat{B}_{x1} \\ \hat{B}_{y1} \end{pmatrix} = 0. \tag{15}$$

Let us now determine the determinant of this matrix M using the standard method and set $\det\mathsf{M}=0$, i.e.,

$$-\mathrm{i}k\frac{B_0}{\mu_0\rho_0}\det\begin{pmatrix} \sigma & -2\Omega & -\mathrm{i}k\frac{B_0}{\mu_0\rho_0} \\ -\mathrm{i}kB_0 & 0 & \sigma \\ 0 & -\mathrm{i}kB_0 & -S \end{pmatrix} + \sigma\det\begin{pmatrix} \sigma & -2\Omega & -\mathrm{i}k\frac{B_0}{\mu_0\rho_0} \\ S+2\Omega & \sigma & 0 \\ -\mathrm{i}kB_0 & 0 & \sigma \end{pmatrix} = 0, \tag{16}$$

i.e.,

$$-\mathrm{i}k\frac{B_0}{\mu_0\rho_0}\left[\mathrm{i}k\frac{B_0}{\mu_0\rho_0}k^2B_0^2 + \mathrm{i}kB_0\sigma^2 + \mathrm{i}kB_0\,2\Omega S\right] + \sigma\left[\sigma^3 + \sigma k^2\frac{B_0^2}{\mu_0\rho_0} + 2\Omega(S+2\Omega)\sigma\right] = 0. \tag{17}$$

It is convenient to define the Alfvén frequency, $\omega_\mathrm{A} = kB_0/\sqrt{\mu_0\rho_0}$, so we have

$$\omega_\mathrm{A}^2(\omega_\mathrm{A}^2+\sigma^2+2\Omega S) + \sigma^2\left[\sigma^2+\omega_\mathrm{A}^2+2\Omega(S+2\Omega)\right] = 0, \tag{18}$$

or

$$\sigma^4 + 2\sigma^2[\omega_\mathrm{A}^2+\Omega(S+2\Omega)] + \omega_\mathrm{A}^2(\omega_\mathrm{A}^2+2\Omega S) = 0. \tag{19}$$

This is a biquadratic equation with the solution

$$\sigma_\pm^2 = -[\omega_\mathrm{A}^2+\Omega(S+2\Omega)] \pm \sqrt{[\omega_\mathrm{A}^2+\Omega(S+2\Omega)]^2 - \omega_\mathrm{A}^2(\omega_\mathrm{A}^2+2\Omega S)}, \tag{20}$$

or

$$\sigma_\pm^2 = -[\omega_\mathrm{A}^2+\Omega(S+2\Omega)] \pm \sqrt{2\omega_\mathrm{A}^2\Omega(S+2\Omega) + \Omega^2(S+2\Omega)^2 - 2\omega_\mathrm{A}^2\Omega S}, \tag{21}$$

and thus

$$\sigma_\pm^2 = -[\omega_\mathrm{A}^2+\Omega(S+2\Omega)] \pm \Omega\sqrt{4\omega_\mathrm{A}^2+(S+2\Omega)^2}. \tag{22}$$

3 Discussion of the solution branches

Let us first of all note that for $\Omega=S=0$, there are two degenerate solutions, each having negative σ^2 and thus $\sigma=\pm\mathrm{i}\omega_\mathrm{A}$. These correspond to slow magnetosonic waves (lower sign) and Alfvén waves (upper sign). This degeneracy is lifted when $\boldsymbol{k}$ and $\boldsymbol{B}_0$ are not parallel to each other. However, even in the present case where both are parallel, the degeneracy is lifted once we turn on rotation. This is shown in Figure 2, where we plot σ^2 for different values of Ω. The upper branch (upper sign) approaches the ω_A^2 axis while the lower one goes further downward.

Next, consider the case with shear. For negative shear, there is a range of values of ω_A^2 with positive solutions for σ^2, i.e., σ is real, and one solution is positive (unstable) and one is negative (stable). In particular, for the Keplerian case, we have unstable (and nonoscillatory) solutions when $0<\omega_\mathrm{A}^2<3\Omega$.

Note also that the instability is stabilized for strong magnetic fields (large values of B_0), unless the domain is big enough to accommodate sufficiently small values of k.

In an unbounded domain, negative values of k^2 are unphysical, because the eigenfunctions would blow up at infinity. This is however not the case in a bounded domain.

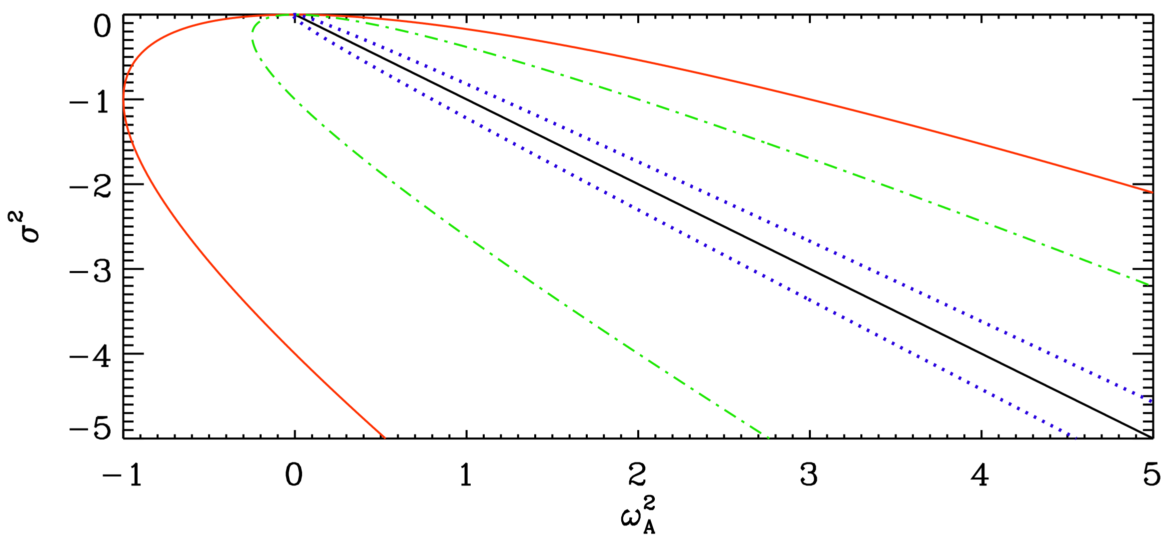

Figure 2: Dispersion relation for $S = 0$ and $\Omega = 0$ (black), 0.1 (dotted blue), 0.5 (dot-dashed green), and 1 (solid red).

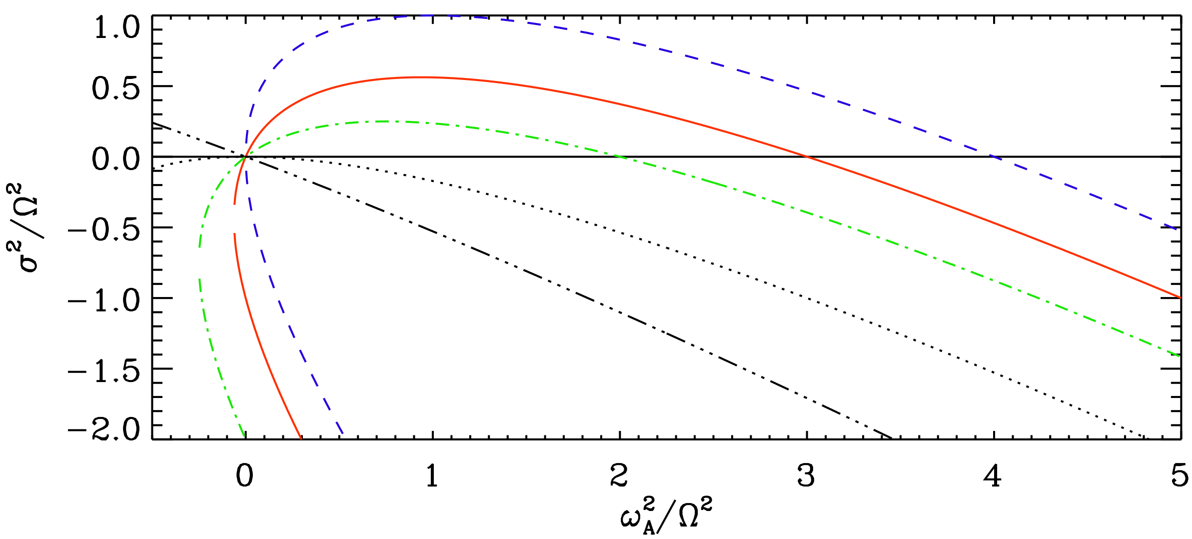

Figure 3: Dispersion relation for $S/\Omega = -3/2$ (Keplerian case, solid red), compared with $S/\Omega = -2$ (dashed blue), -1 (dash-dotted green), 0 (dotted black), and +2 (triple-dash dotted).

References

Balbus, S. A., & Hawley, J. F., "A powerful local shear instability in weakly magnetized disks. I. Linear analysis," *Astrophys. J.* **376**, 214-222 (1991).

Velikhov, E. P., "Stability of an ideally conducting liquid flowing between cylinders rotating in a magnetic field," *Sov. Phys. JETP* **36**, 1398-1404 (1959). (Vol. 9, p. 995 in English translation)

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