Handout 4: Rayleigh–Bénard problem (Part II)

Most of the effort in comparing with laboratory measurements went into the treatment of suitable boundary conditions. Let us consider here the no-slip condition, i.e.,

$$u_x = u_y = u_z = 0 \tag{1}$$

Owing to $\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$, this implies $u_{z,z} = 0$, in addition to $u_z = 0$. Such a function can no longer be represented by simple sine and cosine series. Let us discuss here consequences for the stability analysis.

1 Normal mode analysis

One usually speaks of normal mode analysis, when the eigenfunction is decomposed into a *complete* set of functions. For the time being, we continue using a Fourier decomposition, but now only in the horizontal direction, so we set $u_{1z} = \hat{u}_{1z}(z)\, e^{\sigma t + \mathrm{i}\boldsymbol{k}_\perp\cdot\boldsymbol{x}_\perp}$, Let us inset this into Eq. (15) from Handout 3, i.e.,

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)\left(\mathrm{Pr}\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2 u_{z1} = \mathrm{Ra}\,\nabla_\perp^2 u_{z1}. \tag{2}$$

This yields

$$\left(\sigma + k_\perp^2 - D^2\right)\left(\mathrm{Pr}\,\sigma + k_\perp^2 - D^2\right)\left(k_\perp^2 - D^2\right)\hat{u}_{1z}(z) = \mathrm{Ra}\,k_\perp^2\,\hat{u}_{1z}(z), \tag{3}$$

where $D = \partial/\partial z$ has been introduced as a shorthand; this is not to be confused with the advective derivative used earlier.

Another trick that can be invoked is what is called the *principle of the exchange of stabilities*, which really just means that σ is real and that the marginal states are characterized by $\sigma = 0$. We discussed this in Handout 3, but didn't talk about exchange of stabilities. Chandrasekhar (1961) talks a lot about it and gives in his Section 11 a general proof of this for Rayleigh-Bénard convection in the absence of rotation. In the presence of rotation, however, the principle of the exchange of stabilities is not valid.

Thus, putting $\sigma = 0$ in Equation (3), and multiplying by -1 (so the coefficient in front of the highest derivative is positive) we have

$$\left(D^2 - k_\perp^2\right)^3\hat{u}_{1z}(z) = -\mathrm{Ra}\,k_\perp^2\,\hat{u}_{1z}(z). \tag{4}$$

Note that this equation, which describes only the onset of convection, is independent of Pr. We have seen this before where the *marginal* stability condition for stress-free boundary conditions was independent of Pr.

The general solution can now we written as a superposition of solutions of the form

$$\hat{u}_{1z}(z) = \sum_{\pm i=1}^{3} A_i e^{q_i z} \tag{5}$$

with, in general, complex values of q_i. Inserting this into Equation (4) yields

$$\left(q_i^2 - k_\perp^2\right)^3 = -\mathrm{Ra}\,k_\perp^2 \tag{6}$$

for $i = 1$, 2, and 3. To solve this equation, we need to find the three roots of this equation. The footnote[1]

[1] To find the three roots of $(-1)^{1/3}$, it is useful to represent -1 in the form $-1 = e^{i\pi}$. The three solutions are then

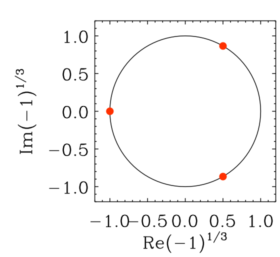
$$e^{+i\pi/3} = \frac{1}{2} + \frac{\mathrm{i}}{2}\sqrt{3}, \tag{7}$$

$$e^{-i\pi/3} = \frac{1}{2} - \frac{\mathrm{i}}{2}\sqrt{3}, \tag{8}$$

and

$$e^{i\pi} = -1. \tag{9}$$

Likewise, if we wanted to find the roots of $(-1)^{1/5}$, for example, they would be given by $e^{\pm \mathrm{i}\pi/5} = \cos\pi/5 \pm \sin\pi/5$, $e^{\pm 3\mathrm{i}\pi/5} = \cos 3\pi/5 \pm \sin 3\pi/5$, and, again, $e^{5\mathrm{i}\pi/5} = -1$.



is a reminder of how you do this. With these preparations, we can now write

$$q_i^2 - k_\perp^2 = \mathrm{Ra}^{1/3}\, k_\perp^{2/3} \times \begin{cases} -1 & \text{for } i=1 \\ \frac{1}{2} + \frac{\mathrm{i}}{2}\sqrt{3} & \text{for } i=2 \\ \frac{1}{2} - \frac{\mathrm{i}}{2}\sqrt{3} & \text{for } i=3 \end{cases} \tag{10}$$

for the three roots of q_i^2. To find all six roots of q_i, we begin with the simplest case, i.e.,

$$q_{\pm 1} = \pm\sqrt{k_\perp^2 - \mathrm{Ra}^{1/3}\, k_\perp^{2/3}} = \pm\mathrm{i}\sqrt{\mathrm{Ra}^{1/3}\, k_\perp^{2/3} - k_\perp^2} = \pm\mathrm{i}k_\perp\sqrt{(\mathrm{Ra}/k_\perp^4)^{1/3} - 1}. \tag{11}$$

Next, we have

$$q_{\pm 2} = \pm\sqrt{k_\perp^2 + \mathrm{Ra}^{1/3}\, k_\perp^{2/3}\, \left(\tfrac{1}{2} + \tfrac{\mathrm{i}}{2}\sqrt{3}\right)} = \pm k_\perp\sqrt{1 + \tfrac{1}{2}\left(\mathrm{Ra}/k_\perp^4\right)^{1/3}\, \left(1 + \mathrm{i}\sqrt{3}\right)} \tag{12}$$

and finally, $q_{\pm 3}$ is just given by the complex conjugate of $q_{\pm 2}$, i.e.,

$$q_{\pm 3} = q_{\pm 2}^*. \tag{13}$$

To construct the final solution and to determine the critical excitation condition, we need to invoke boundary conditions. In addition to those discussed in the preamble, i.e., $\hat{u}_{1z} = D\hat{u}_{1z} = 0$, we still have the condition $\hat{T} = 0$, which can be expressed in terms of $\hat{u}_{1z}$ using Eq. (10) of Handout 3, which reduces to

$$\left(D^2 - k_\perp^2\right)^2 \hat{u}_{1z}(z) = 0 \tag{14}$$

For each of the three pairs, the functions can be readily combined into a function that is symmetric around 0 by

$$\hat{u}_{1z}(z) = \sum_{\pm i=1}^{3} A_i e^{q_i z} = \sum_{i=1}^{3} A_i \left(e^{q_i z} + e^{-q_i z}\right) = 2\sum_{i=1}^{3} A_i \cosh q_i z. \tag{15}$$

To obey the boundary condition $\hat{u}_{1z}(\pm 1/2) = 0$, we have to require that

$$\sum_{i=1}^{3} \cosh q_i/2 = 0. \tag{16}$$

This is one equation for the three unknowns A_i for $i = 1$, 2, and 3. Next, to obey the boundary condition $D\hat{u}_{1z}(\pm 1/2) = 0$, we have to require that

$$\sum_{i=1}^{3} \sinh q_i/2 = 0. \tag{17}$$

Finally, to obey the boundary condition $(D^2 - k_\perp^2)^2 \hat{u}_{1z}(\pm 1/2) = 0$, we have to require that

$$\sum_{i=1}^{3} \left(q_i^2 - k_\perp^2\right)^2 \cosh q_i/2 = 0. \tag{18}$$

We now have 3 equations for the three unknowns A_i for $i = 1$, 2, and 3. This leads to a 3×3 matrix equation, where the eigenvector is given by (A_1, A_2, A_3) and the matrix is a function of Ra and $k_\perp^2$. The determinant of this matrix must vanish, which then results in a function $\mathrm{Ra} = \mathrm{Ra}(k_\perp^2)$; see Fig. 11.10 of KCD. The smallest value of Ra is reached at $k_\perp = 3.12$ and gives $\mathrm{Ra}(k_\perp) = 1708$.

References

Chandrasekhar, S. *Hydrodynamic and Hydromagnetic Stability.* Dover Publications, New York (1961).

$Header: /var/cvs/brandenb/tex/teach/ASTR_5410/4_Rayleigh_Benard_II/notes.tex,v 1.4 2016/08/31 16:44:03 brandenb Exp $