Handout 4b: Rayleigh–Bénard problem (Part IIb)

Let us recall that the three boundary conditions yield[1]

$$\sum_{i=1}^{3} \cosh q_i/2 = 0. \tag{1}$$

$$\sum_{i=1}^{3} q_i \sinh q_i/2 = 0. \tag{2}$$

$$\sum_{i=1}^{3} \left(q_i^2 - k_\perp^2\right)^2 \cosh q_i/2 = 0. \tag{3}$$

We write this in matrix form, which then yields

$$\det \begin{pmatrix} \cosh q_1/2 & \cosh q_2/2 & \cosh q_2/2 \\ q_1 \sinh q_1/2 & q_2 \sinh q_2/2 & q_3 \sinh q_3/2 \\ \left(q_1^2 - k_\perp^2\right)^2 \cosh q_1/2 & \left(q_2^2 - k_\perp^2\right)^2 \cosh q_2/2 & \left(q_3^2 - k_\perp^2\right)^2 \cosh q_3/2 \end{pmatrix} = 0 \tag{4}$$

Making use of the fact that q_1 is purely imaginary, and thus defining $q_0 = \mathrm{i}q_1$, and using the fact that $\mathrm{i}\, q_0 \sinh \mathrm{i}\, q_0/2 = -q_0 \sin q_0/2$, we have

$$\det \begin{pmatrix} \cos q_0/2 & \cosh q_2/2 & \cosh q_2/2 \\ -q_0 \sin q_0/2 & q_2 \sinh q_2/2 & q_3 \sinh q_3/2 \\ \left(q_0^2 + k_\perp^2\right)^2 \cos q_0/2 & \left(q_2^2 - k_\perp^2\right)^2 \cosh q_2/2 & \left(q_3^2 - k_\perp^2\right)^2 \cosh q_3/2 \end{pmatrix} = 0 \tag{5}$$

Next, we divide all rows by row 1,

$$\det \begin{pmatrix} 1 & 1 & 1 \\ -q_0 \tan q_0/2 & q_2 \tanh q_2/2 & q_3 \tanh q_3/2 \\ \left(q_0^2 + k_\perp^2\right)^2 & \left(q_2^2 - k_\perp^2\right)^2 & \left(q_3^2 - k_\perp^2\right)^2 \end{pmatrix} = 0 \tag{6}$$

We recall that, using Eq.(10) of Handout 4,

$$q_i^2 - k_\perp^2 = \mathrm{Ra}^{1/3}\, k_\perp^{2/3} \times \begin{cases} -1 & \text{for } i = 1 \\ \frac{1}{2} + \frac{\mathrm{i}}{2}\sqrt{3} & \text{for } i = 2 \\ \frac{1}{2} - \frac{\mathrm{i}}{2}\sqrt{3} & \text{for } i = 3 \end{cases}, \tag{7}$$

so we have

$$\det \begin{pmatrix} 1 & 1 & 1 \\ -q_0 \tan q_0/2 & q_2 \tanh q_2/2 & q_3 \tanh q_3/2 \\ 1 & \frac{1}{2}(\mathrm{i}\sqrt{3} - 1) & -\frac{1}{2}(\mathrm{i}\sqrt{3} + 1) \end{pmatrix} \tag{8}$$

Subtracting now row 1 from from 3, recalling that $q_3 = q_2^*$, and dividing by $-\sqrt{3}/2$ we have

$$\det \begin{pmatrix} 1 & 1 & 1 \\ -q_0 \tan q_0/2 & q_2 \tanh q_2/2 & q_2^* \tanh q_2^*/2 \\ 0 & \sqrt{3} - \mathrm{i} & \sqrt{3} + \mathrm{i} \end{pmatrix} \tag{9}$$

This yields

$$\mathrm{Im}\left[(\sqrt{3} + \mathrm{i}) q_2 \tanh q_2\right] + q_0 \tan q_0/2. \tag{10}$$

Instead of solving this equation, we just draw a contour plot; see Figure 1. The smallest value of Ra is reached at $k_\perp = 3.12$ and gives $\mathrm{Ra}(k_\perp) = 1708$.

[1]Note the q_i term in front of sinh, which I forgot last time!

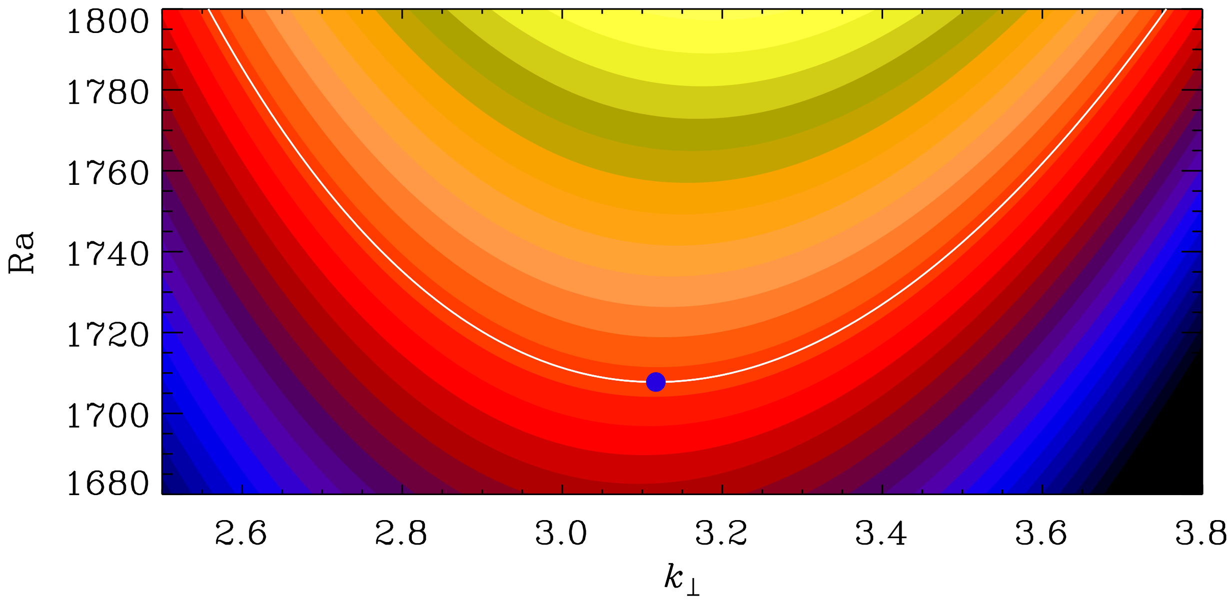

Figure 1: Determinant in the plane Ra versus $k_\perp = 3.12$.

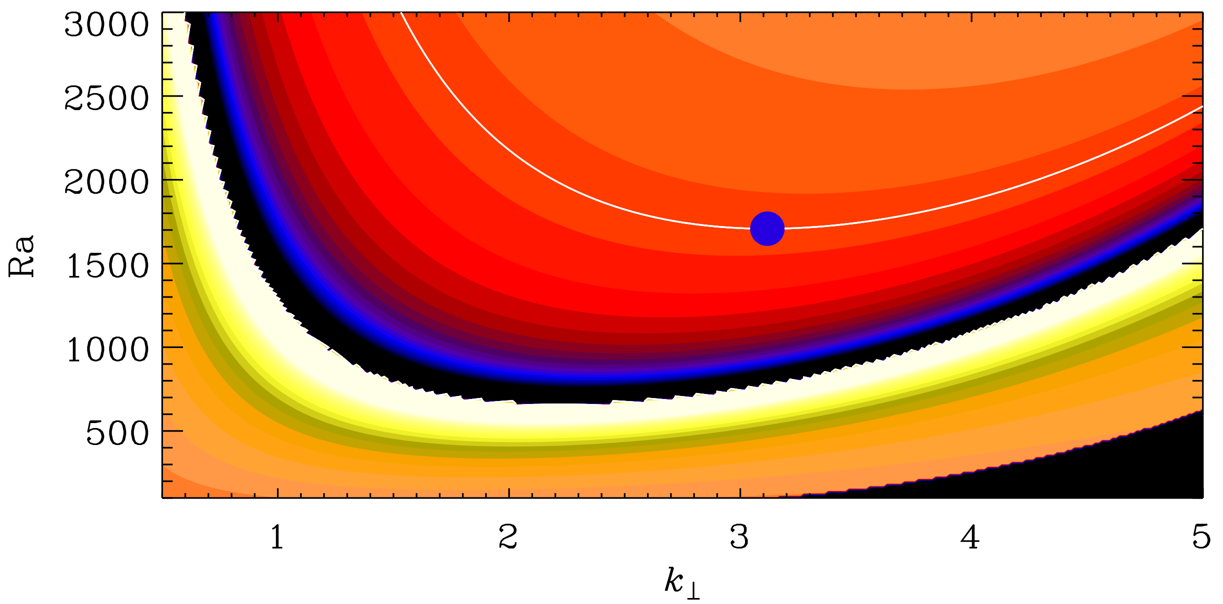

Figure 2: Similar to Figure 1, but for a larger range.

References

Chandrasekhar, S. *Hydrodynamic and Hydromagnetic Stability.* Dover Publications, New York (1961).

$Header: /var/cvs/brandenb/tex/teach/ASTR_5410/4_Rayleigh_Benard_II/notes2.tex,v 1.1 2016/09/02 16:33:24 brandenb Exp $