Handout 5: Rayleigh–Bénard problem (Part III)

1 Weakly nonlinear analysis

You may have noticed that the eigenfunctions discussed above all correspond to rolls and not to hexagonal cells, which is what is usually seen in laboratory experiments. Also, of course, the value of the Prandtl number didn't even enter in the stability analysis, except for linear growth rates. However, the growth rates from the linear theory are only of limited usefulness beyond the onset of instability at $\mathrm{Ra} = \mathrm{Ra}_{\mathrm{crit}}$. To go beyond the linear stability analysis, one has to continue the analysis to higher orders, beyond T_0, T_1, and $\boldsymbol{u}_1$. This procedure was first explored in a seminar paper by Schlüter et al. (1965), and it has subsequently been used in many other fields, such as the theory of the nonlinear development of the laser instability, as well as in biophysics, where pattern formation plays is an important problem. For these broader applications and a pedagogic introduction to this topic, see the review and text book by Haken (1975, 1983).

For compacter notation, it is convenient to introduce the state vector $\boldsymbol{q} = (\boldsymbol{u}, T, P/\rho_0)$. The ansatz for the nonlinear solution to the fully nonlinear equations assumed to be

$$\boldsymbol{q}(\boldsymbol{x}, t) = A_1(t)\boldsymbol{q}^{(1)}(\boldsymbol{x}) + A_2(t)\boldsymbol{q}^{(2)}(\boldsymbol{x}) + A_3(t)\boldsymbol{q}^{(3)}(\boldsymbol{x}) + ... \tag{1}$$

where the $\boldsymbol{q}^{(i)}(\boldsymbol{x})$ are the eigenvectors to the linear problem at $\mathrm{Ra} = \mathrm{Ra}_{\mathrm{crit}}$. These eigenvectors form a complete set, so all nonlinear solutions at $\mathrm{Ra} > \mathrm{Ra}_{\mathrm{crit}}$ can be expanded in terms of these eigenfunctions. Consider now the nonlinear equations in the form

$$\frac{\partial q}{\partial t} = \mathsf{L}\boldsymbol{q} + \mathsf{N}(\boldsymbol{q}). \tag{2}$$

It is convenient to write the $\boldsymbol{q}$ as ket-vectors, $|\boldsymbol{q}\rangle$, and define adjoint eigenvectors, which are solutions of the adjoint problem $\partial\langle\boldsymbol{q}|/\partial t = \langle\boldsymbol{q}|\mathsf{L}$. These eigenvectors are normalized such that $\langle\boldsymbol{q}^{(i)}|\boldsymbol{q}^{(j)}\rangle = \delta_{ij}$. Inserting now Equation (1) into Equation (2) yields the following set of equations:

$$\frac{\mathrm{d}\xi_l}{\mathrm{d}t} = \sigma_l\xi_l - A_{ll'l''}\xi_{l'}\xi_{l''} \tag{3}$$

for all eigenvalues σ_l, where the index l characterizes the eigenstates for different wavenumbers $(k_\perp, k_z)$. Here, the $A_{ll'l''}$ are known coefficients. Since we are still in the weakly nonlinear regime, the higher modes are not yet excited and so σ_l for $l > 1$ will be negative. Furthermore, since the time evolution is slow we can neglect $\mathrm{d}\xi_2/\mathrm{d}t$ compared with $\sigma_2\xi_2$ and solve Equation (3) for $l = 2$.

2 Form of L and eigenvectors $\boldsymbol{q}_l$

It is convenient to return to the original linearized equations in the form

$$\frac{\partial\boldsymbol{u}_1}{\partial t} = -\frac{1}{\rho_0}\boldsymbol{\nabla}P_1 + \alpha T_1 g\hat{\boldsymbol{z}} + \nu\nabla^2\boldsymbol{u}_1 \tag{4}$$

$$\frac{\partial T_1}{\partial t} - \beta u_{z1} = \kappa\nabla^2 T_1. \tag{5}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u}_1 = 0 \tag{6}$$

and write the matrix equation in the form $\partial_t\mathsf{S}\boldsymbol{q} = \mathsf{L}\boldsymbol{q}$ with

$$\mathsf{S} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathsf{L} = \begin{pmatrix} \nu\nabla^2 & 0 & 0 & 0 & -\partial_x \\ 0 & \nu\nabla^2 & 0 & 0 & -\partial_y \\ 0 & 0 & \nu\nabla^2 & \alpha g & -\partial_z \\ 0 & 0 & \beta & \kappa\nabla^2 & 0 \\ \partial_x & \partial_y & \partial_z & 0 & 0 \end{pmatrix}, \quad \boldsymbol{q} = \begin{pmatrix} u_{1x} \\ u_{1y} \\ u_{1z} \\ T_1 \\ P_1/\rho_0 \end{pmatrix} \tag{7}$$

It is possible to rewrite the matrix in hermitian form

$$\mathsf{S}=\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&\mathrm{Pr}&0\\0&0&0&0&0\end{pmatrix},\quad \mathsf{L}=\begin{pmatrix}\nabla^2&0&0&0&-\partial_x\\0&\nabla^2&0&0&-\partial_y\\0&0&\nabla^2&\mathrm{Ra}^{1/2}&-\partial_z\\0&0&\mathrm{Ra}^{1/2}&\nabla^2&0\\\partial_x&\partial_y&\partial_z&0&0\end{pmatrix},\quad \boldsymbol{q}=\begin{pmatrix}u_{1x}\\u_{1y}\\u_{1z}\\T_1\\P_1/\rho_0\end{pmatrix} \tag{8}$$

where the eigenvector has the form

$$\boldsymbol{q}_l=\begin{pmatrix}q_1\cos l\pi z\\q_2\cos l\pi z\\q_3\sin l\pi z\\q_4\sin l\pi z\\q_5\cos l\pi z\end{pmatrix}e^{\mathrm{i}\boldsymbol{k}_\perp\cdot\boldsymbol{x}_\perp} \tag{9}$$

where $l=1,\ 2,\ ...,$ are integers characterizing higher eigenmodes, and the eigenfrequency for the most unstable mode was calculated in lecture 3 to be

$$\sigma_l=-\frac{1+\mathrm{Pr}}{2\,\mathrm{Pr}}+\sqrt{\frac{(1+\mathrm{Pr})^2}{4\,\mathrm{Pr}^2}-\frac{(k_\perp^2+l^2\pi^2)^2}{\mathrm{Pr}}+\frac{\mathrm{Ra}}{\mathrm{Pr}}\,\frac{k_\perp^2}{k_\perp^2+l^2\pi^2}}. \tag{10}$$

The nonlinearity is given by

$$\mathsf{N}(\boldsymbol{q})=-\sum_{i=1}^{3}q_i\nabla_i\mathsf{S}\boldsymbol{q}. \tag{11}$$

It turns out that

$$\frac{\partial\xi_1}{\partial t}=\frac{3}{2}\pi^2\frac{\mathrm{Pr}}{\mathrm{Pr}+1}\,\frac{\mathrm{Ra}-\mathrm{Ra}_{\mathrm{crit}}}{\mathrm{Ra}}\,\xi_1+\frac{2\sqrt{2P}}{9\pi^2(\mathrm{Pr}+1)}\,\xi_1\xi_2 \tag{12}$$

and

$$\frac{\partial\xi_2}{\partial t}=-4\pi^2\,\xi_2-\frac{2\sqrt{2P}}{9\pi^2(\mathrm{Pr}+1)}\,\xi_1^2\xi_1\xi_2 \tag{13}$$

By invoking the principle of the elimination of rapidly adjusting variables, we finally arrive at

$$\frac{\partial\xi_1}{\partial t}=\frac{3}{2}\pi^2\frac{\mathrm{Pr}}{\mathrm{Pr}+1}\,\frac{\mathrm{Ra}-\mathrm{Ra}_{\mathrm{crit}}}{\mathrm{Ra}}\,\xi_1-\frac{P}{6\pi^2\mathrm{Ra}_{\mathrm{crit}}(\mathrm{Pr}+1)^2}\,\xi_1^3. \tag{14}$$

Thus, the solution has a stable fixed point (negative prefactor) and the bifurcation is supercritical.

References

Haken, H., “Cooperative phenomena in systems far from thermal equilibrium and in nonphysical systems,” *Rev. Mod. Phys.* **47**, 67-121 (1975).

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