# Handout 6: Double-diffusive instability

broad range of applications sea water massive stars in which the burning of hydrogen into helium has led to a stabilizing gradient of the mean molecular weight.

## 1 Governing equations

At the technical level, the double-diffusive instability is an extension of the Rayleigh-Bénard problem in that the density is now a function of not only temperature, but also the concentration of salinity (in the ocean) or helium (in deeper layers of a star). Thus, the equation of state for $\rho$ includes now an extra term for this concentration and reads

$$\rho = \rho_{00}\left[1 - \alpha_T(T - T_{00}) + \alpha_C(C - C_{00})\right] \tag{1}$$

Thus, the momentum equation becomes

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\nabla^2 u_{z1} = \alpha_T g \nabla_\perp^2 T_1 - \alpha_C g \nabla_\perp^2 C_1, \tag{2}$$

and for temperature and concentration we have respectively

$$\left(\frac{\partial}{\partial t} - \kappa_T\nabla^2\right) T_1 = \beta_T u_{z1}, \quad \left(\frac{\partial}{\partial t} - \kappa_C\nabla^2\right) C_1 = \beta_C u_{z1}. \tag{3}$$

Applying the operators of the left-hand sides of Equation (3) to Equation (2), we have

$$\left(\frac{\partial}{\partial t} - \kappa_T\nabla^2\right)\left(\frac{\partial}{\partial t} - \kappa_C\nabla^2\right)\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\nabla^2 u_{z1} = \left[\left(\frac{\partial}{\partial t} - \kappa_C\nabla^2\right)\alpha_T\beta_T - \left(\frac{\partial}{\partial t} - \kappa_T\nabla^2\right)\alpha_C\beta_C\right] g\nabla_\perp^2 u_{z1}. \tag{4}$$

Assuming solutions to be of the form $u_{z1} = \hat{u}_{z1}(z)\, e^{\sigma t + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$, we have

$$\left(\sigma + \kappa_T k^2\right)\left(\sigma + \kappa_C k^2\right)\left(\sigma + \nu k^2\right) k^2 = \left[\left(\sigma + \kappa_C k^2\right)\alpha_T\beta_T - \left(\sigma + \kappa_T k^2\right)\alpha_C\beta_C\right] g k_\perp^2. \tag{5}$$

If the principle of the exchange of stabilities were applicable, we would have

$$\kappa_T\kappa_C\nu k^6 = \left(\kappa_C\alpha_T\beta_T - \kappa_T\alpha_C\beta_C\right) g k_\perp^2. \tag{6}$$

Thus, the condition of marginal stability can be written in the form

$$\frac{k^6}{k_\perp^2} = \frac{\alpha_T\beta_T\, g}{\kappa_T\nu} - \frac{\alpha_C\beta_C\, g}{\kappa_C\nu}. \tag{7}$$

Thus, we see that the difference between two suitably defined Rayleigh numbers has to be big enough. Moreover, we can envisage two quite different situations:

(i) $\beta_T$ is large (larger than by the usual marginal stability criterion without salinity), but $\beta_C$ is also large so that the system is being stabilized. In astrophysics, the resulting state is called *semi-convection*.

(ii) $\beta_T$ is negative (or at least smaller than by the usual marginal stability criterion without salinity), but $\beta_C$ is now also negative so that the system is being destabilized. In oceanographics, the resulting state is called *thermohaline convection* and was discovered by Stern (1960).

## 2 Dispersion relation

Let us now work out the dispersion relation:

$$\left(\sigma+\kappa_T k^2\right)\left(\sigma+\kappa_C k^2\right)\left(\sigma+\nu k^2\right)-\left[\left(\sigma+\kappa_C k^2\right)\alpha_T\beta_T-\left(\sigma+\kappa_T k^2\right)\alpha_C\beta_C\right]\,g\,\frac{k_\perp^2}{k^2}=0. \tag{8}$$

Thus,

$$\sigma^3+\sigma^2(\kappa_T+\kappa_C+\nu)k^2+\sigma\left[(\kappa_T\kappa_C+\kappa_C\nu+\nu\kappa_T)k^4+(\alpha_C\beta_C-\alpha_T\beta_T)g\,\frac{k_\perp^2}{k^2}\right]$$
$$+\kappa_T\kappa_C\nu k^6+(\alpha_C\beta_C\kappa_T-\alpha_T\beta_T\kappa_C)g\,k_\perp^2=0$$

It may be more intuitive to define $\alpha_T\beta_T g=-N_T^2$ and $\alpha_C\beta_C g=-N_C^2$, so that

$$\sigma^3+\sigma^2(\kappa_T+\kappa_C+\nu)k^2+\sigma\left[(\kappa_T\kappa_C+\kappa_C\nu+\nu\kappa_T)k^4+(N_T^2-N_C^2)\frac{k_\perp^2}{k^2}\right]$$
$$+\kappa_T\kappa_C\nu k^6+(N_T^2\kappa_C-N_C^2\kappa_T)k_\perp^2=0$$

Let us now introduce nondimensional units by defining $\sigma/\nu k^2\to\sigma$, $\kappa_T/\nu\to\kappa_T$, $\kappa_C/\nu\to\kappa_C$, $N_T^2/\nu^2k^4\to N_T^2$, $N_C^2/\nu^2k^4\to N_C^2$, and $k_\perp^2/k^2\to\kappa_\perp^2$

$$\sigma^3+\sigma^2(\kappa_T+\kappa_C+1)+\sigma\left[(\kappa_T\kappa_C+\kappa_C+\kappa_T)+(N_T^2-N_C^2)k_\perp^2\right]$$
$$+\kappa_T\kappa_C+(N_T^2\kappa_C-N_C^2\kappa_T)k_\perp^2=0.$$

This equation is now dimensionless, but we still have five parameters to vary! In Figure 1 the dispersion relation is plotted for the case of an oscillatory onset of convection (so-called semiconvection) for $N_T^2=-1.5$ (unstable) and $N_C^2=-1$ (stabilizing), using $k_\perp^2/k^2=0.5$.

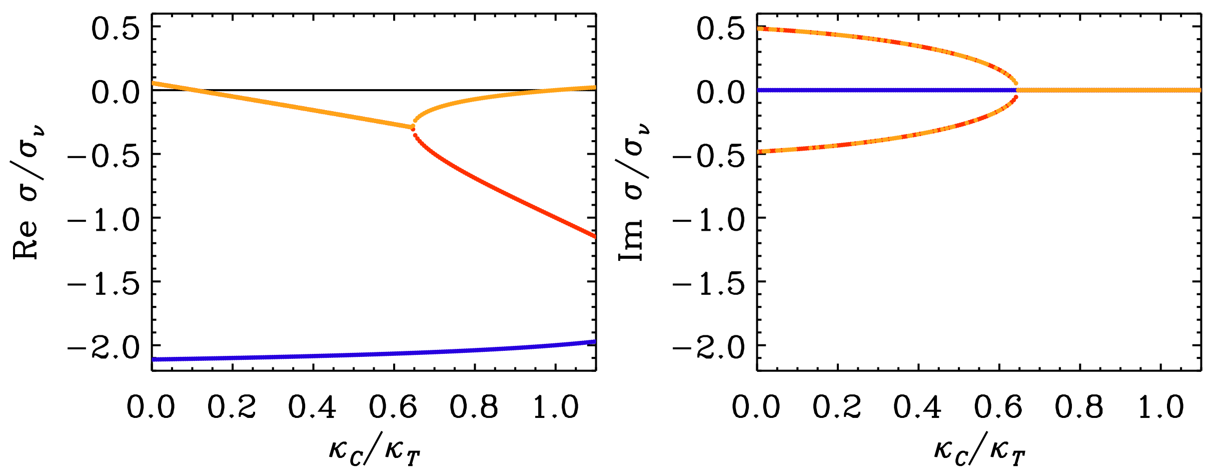


Figure 1: Real and imaginary parts of $\sigma$ for $N_T^2=-1.5$, $N_C^2=-1$, for $k_\perp^2/k^2=0.5$. Note that $\mathrm{Re}\sigma>0$ (unstable) for $\kappa_C/\kappa_T\lesssim 0.1$. At the same time, $\mathrm{Im}\sigma\neq 0$ (oscillatory).

## References


Stern, M. E., "The salt-fountain and thermohaline convection," *Tellus* **12**, 172-175 (1960).



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