Handout 6b: Double-diffusive instability

In the previous handout (Handout 6),[1] we showed numerical solutions of σ in units of $\sigma_\nu = \nu k^2$. This is useful for a first screening of the results and to find parameter values of where something interesting (such as oscillatory solutions) happens. Eventually one also wants to see the k dependence. This will be done here in the present Handout 6b. There will also be some general comments on how to do this.

1 Solving the dispersion relation numerically

In Handout 6, we derived the dispersion relation in the form

$$a_3\sigma^3 + a_2\sigma^2 + a_1\sigma + a_0 = 0 \tag{1}$$

with

$$a_3 = 1, \tag{2}$$

$$a_2 = (\kappa_T + \kappa_C + \nu)k^2, \tag{3}$$

$$a_1 = \left[(\kappa_T\kappa_C + \kappa_C\nu + \nu\kappa_T)k^4 + (N_T^2 - N_C^2)\frac{k_\perp^2}{k^2}\right], \tag{4}$$

$$a_0 = \kappa_T\kappa_C\nu k^6 + (N_T^2\kappa_C - N_C^2\kappa_T)k_\perp^2 = 0, \tag{5}$$

where $\alpha_T\beta_T g = -N_T^2$ and $\alpha_C\beta_C g = -N_C^2$ correspond to the buoyancy frequencies if only one of the two gradients were different from zero. These quantities are also related to corresponding Rayleigh numbers, $\mathrm{Ra}_T = -N_T^2 d^4/\nu\kappa_T$ (which is the standard Rayleigh number, and $\mathrm{Ra}_C = -N_C^2 d^4/\nu\kappa_T$ (which is the Rayleigh number for salinity, as defined in the literature.[2] During the lecture, the real and imaginary parts of σ/σ_ν for $N_T^2 = -3$, $N_C^2 = -1$, and $k_\perp^2/k^2 = 0.5$ will be shown. We will see that $\mathrm{Re}\sigma > 0$ (unstable) for $\kappa_C/\kappa_T \lesssim 0.1$. At the same time, $\mathrm{Im}\sigma \neq 0$, so the onset of instability occurred in an oscillatory fashion. This also implies that the principle of the exchange of stabilities is not valid in this case.

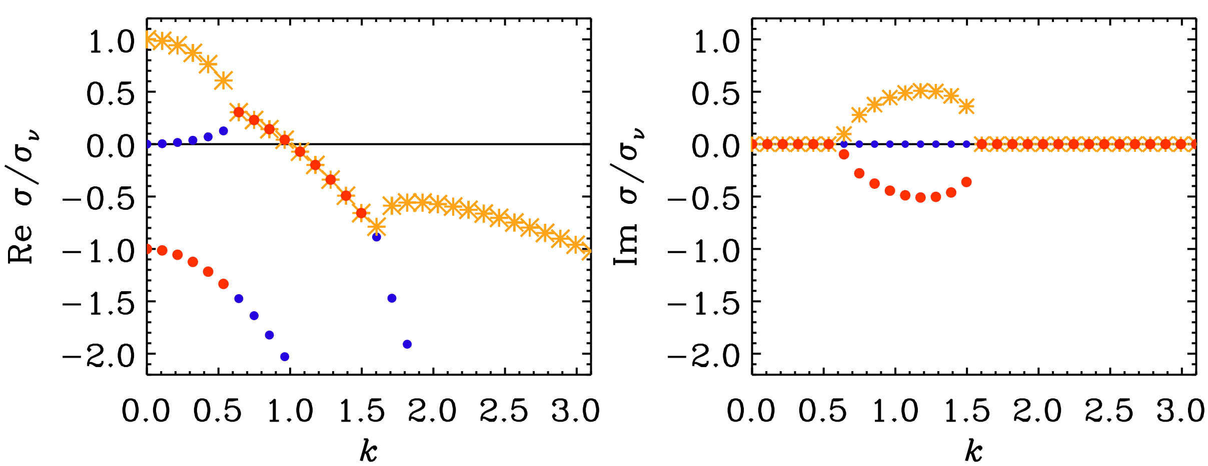

Figure 1: Real and imaginary parts of $\sigma(k)$ for $N_T^2 = -3$, $N_C^2 = -1$, for $k_\perp^2/k^2 = 0.5$. Note that $\mathrm{Re}\sigma > 0$ (unstable) for $k < 1$, where $\mathrm{Im}\sigma \neq 0$ (corresponding to oscillatory solutions).

[1] Owing to a typo in the plotting routine, the y-title said σ_ρ, but it should have read σ_ν.

[2] A division by $\nu\kappa_C$ instead would have made more sense, but for historical reasons the division by $\nu\kappa_T$ has been kept.

To solve the cubic equation, one can just use a library routine, e.g. `fz_roots` in idl. Here we call it for a certain number of k values, keeping however $k_\perp^2/k^2 = \text{const}$, and sort the resulting values by increasing imaginary part and color the data points differently; see Figure 1.

As is common in Rayleigh-Bénard convection, instability occurs only for small values of k (here $k < 1$). Here the imaginary part is nonvanishing (red and orange data points). For $k < 0.7$ the imaginary part vanishes, but this does not necessarily imply anything about the solution in the nonlinear regime.

Figure 2 shows the corresponding case for $N_T^2 = -3$, $N_C^2 = -1$, which is the case where both T and C gradients are positive, so the temperature is stably stratified and salinity unstably. In this case, the onset of instability is non-oscillatory.

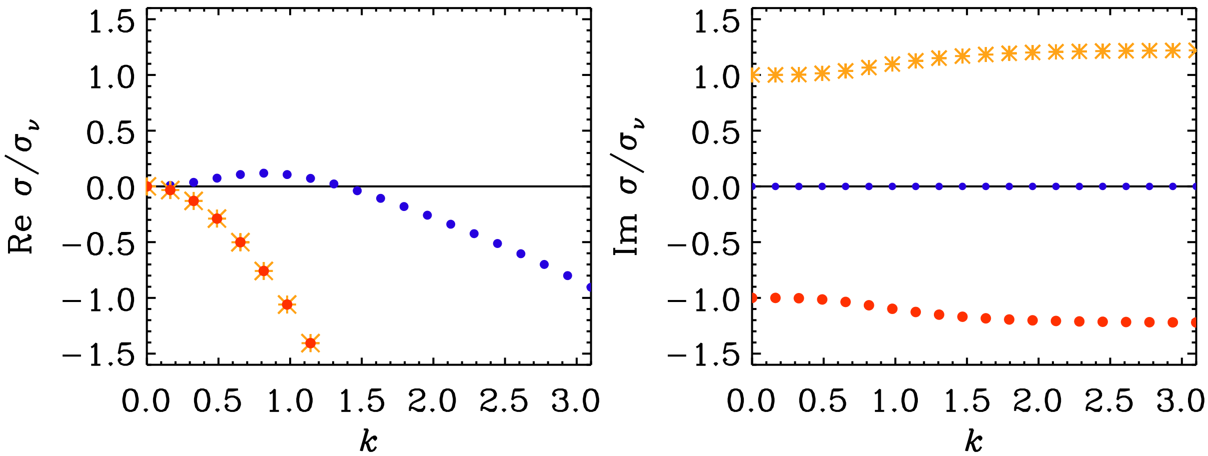

Figure 2: Real and imaginary parts of $\sigma(k)$ for $N_T^2 = -3$, $N_C^2 = -1$, for $k_\perp^2/k^2 = 0.5$. Note that $\text{Re}\sigma > 0$ (unstable) for $k < 1$, where $\text{Im}\sigma \neq 0$ (corresponding to oscillatory solutions).

References

Stern, M. E., "The salt-fountain and thermohaline convection," *Tellus* **12**, 172-175 (1960).

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