Handout 7: Inflection point instability

We consider the stability of a parallel shear flow $\boldsymbol{U} = \hat{\boldsymbol{y}}U(x)$ and write the velocity as $(u_x, U + u_y, 0)$. Recall that the nonlinearity associated with the background flow, $\boldsymbol{U}\cdot\boldsymbol{\nabla}\boldsymbol{U}$, vanishes. To appreciate the essence of the instability, we restrict ourselves to the inviscid case in 2-D.

1 Linearized equations

We consider the incompressible Euler equations in the form

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} - \boldsymbol{\nabla}p, \tag{1}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0. \tag{2}$$

where p corresponds to the reduced pressure, P/ρ_{00}. Linearize around $\boldsymbol{U}$ and write out in component form

$$\frac{\partial u_x}{\partial t} + U\frac{\partial u_x}{\partial y} = -\frac{\partial p}{\partial x} \tag{3}$$

$$\frac{\partial u_y}{\partial t} + U\frac{\partial u_y}{\partial y} + u_x\frac{\partial U}{\partial x} = -\frac{\partial p}{\partial y} \tag{4}$$

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 \tag{5}$$

Assume

$$\begin{aligned} u_x(x,y,t) &= \hat{u}_x(x)\, e^{\mathrm{i}ky+\sigma t} \\ u_y(x,y,t) &= \hat{u}_y(x)\, e^{\mathrm{i}ky+\sigma t} \\ p(x,y,t) &= \hat{p}(x)\, e^{\mathrm{i}ky+\sigma t} \end{aligned} \tag{6}$$

so

$$(\sigma + \mathrm{i}kU)\hat{u}_x = -\hat{p}', \tag{7}$$

$$(\sigma + \mathrm{i}kU)\hat{u}_y + \hat{u}_x U' = -\mathrm{i}k\hat{p}, \tag{8}$$

$$\hat{u}_x' + \mathrm{i}k\hat{u}_y = 0. \tag{9}$$

or in matrix form

$$\sigma\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\hat{u}_x\\ \hat{u}_y\\ \hat{p}\end{pmatrix} = -\begin{pmatrix}\mathrm{i}kU & 0 & \partial_x\\ U' & \mathrm{i}kU & \mathrm{i}k\\ \partial_x & \mathrm{i}k & 0\end{pmatrix}\begin{pmatrix}\hat{u}_x\\ \hat{u}_y\\ \hat{p}\end{pmatrix} \tag{10}$$

Applying impenetrable boundary conditions on $x = \pm L_x/2$ implies $\hat{u}_x = 0$ on both boundaries. (Since there is no viscosity, there is also no viscous stress, and so stress-free boundary conditions cannot be applied.)

2 Numerical treatment

By discretizing in the x direction with n meshpoints, we have a $3n\times 3n$ eigenmatix, where the differential operator, discretized to second order, is written as

$$\partial_x = \begin{pmatrix} 0 & (2\delta_x)^{-1} & ... & ... & ...\\ -(2\delta_x)^{-1} & 0 & (2\delta_x)^{-1} & ... & ...\\ ... & -(2\delta_x)^{-1} & 0 & (2\delta_x)^{-1} & ...\\ ... & ... & ... & ... & ... \end{pmatrix} \tag{11}$$

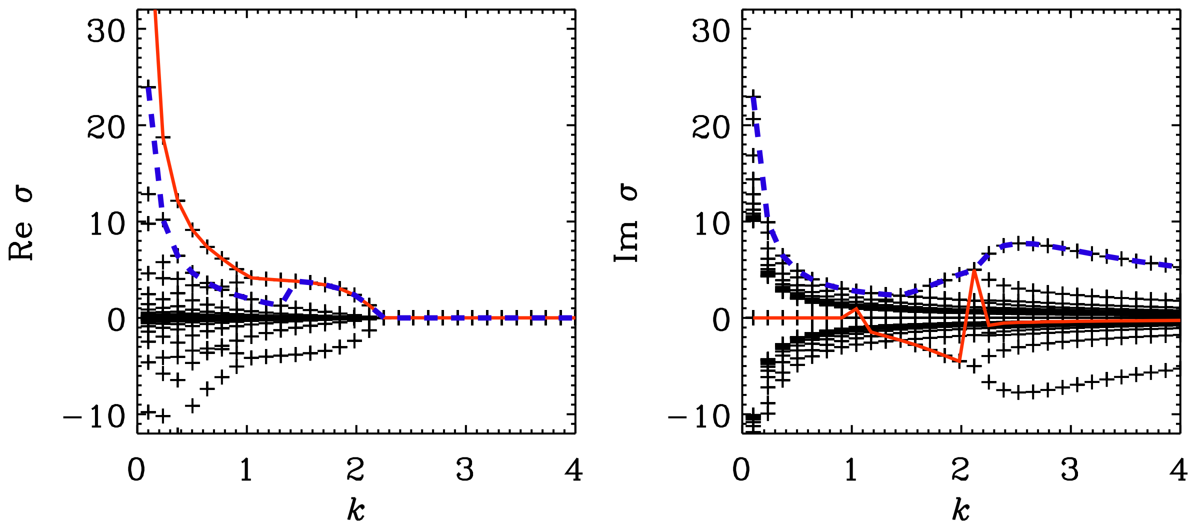

Figure 1: Numerical solutions of the discretized eigenvalue problem using 60 meshpoints.

where δx is the mesh width. It is often convenient to incorporate the boundary condition into the matrix itself, so the boundary points are therefore not part of the matrix. The above matrix is unchanged if the relevant variable has a vanishing value on the boundary. In the case of a vanishing derivative, e.g. for $\hat{p}(x)$ we use a one-sided finite difference formula, so we have

$$\begin{aligned} 0 &= +\hat{p}(x_0) \quad -4\hat{p}(x_1) \quad +3\hat{p}(x_2) \\ (2\delta_x)\,\hat{p}'(x_1) &= -\hat{p}(x_0) \qquad\qquad\qquad +\hat{p}(x_2) \end{aligned} \tag{12}$$

which allows us to eliminate the boundary point, $\hat{p}(x_0)$, so

$$(2\delta_x)\,\hat{p}'(x_1) = -4\hat{p}(x_1) + 4\hat{p}(x_2) \tag{13}$$

or

$$\hat{p}'(x_1) = -2\delta_x^{-1}\hat{p}(x_1) + 2\delta_x^{-1}\hat{p}(x_2) \tag{14}$$

One obtains $3n$ eigenvalues from the $3n \times 3n$ eigenmatix problems. It has to be solved for each value of k. The result is plotted in Figure 1.

3 Analytical treatment

It is convenient to define a stream function so that Equation (9) is automatically obeyed, so

$$\boldsymbol{u} = \boldsymbol{\nabla} \times (\psi\hat{\boldsymbol{z}}) \tag{15}$$

so that

$$u_x = \partial_x\psi, \quad u_y = -\partial_y\psi, \tag{16}$$

or, for $\hat{u}_x$ and $\hat{u}_y$,

$$\hat{u}_x = \mathrm{i}k\hat{\psi}, \quad \hat{u}_y = -\hat{\psi}'. \tag{17}$$

Inserting this into Equations (7) and (8), we have

$$(\sigma + \mathrm{i}kU)\mathrm{i}k\hat{\psi} = -\hat{p}', \tag{18}$$

$$-(\sigma + \mathrm{i}kU)\hat{\psi}' + \mathrm{i}k\hat{\psi}U' = -\mathrm{i}k\hat{p}, \tag{19}$$

Furthermore, it is convenient to work with a wave speed c and write $\sigma = -\mathrm{i}kc$, so that

$$\mathrm{i}k(U-c)\mathrm{i}k\hat{\psi} = -\hat{p}', \tag{20}$$

$$-\mathrm{i}k(U-c)\hat{\psi}' + \mathrm{i}k\hat{\psi}U' = -\mathrm{i}k\hat{p}, \tag{21}$$

In the last equation, the $\mathrm{i}k$ term cancels, so

$$-k^2(U-c)\hat{\psi} = -\hat{p}', \tag{22}$$

$$-(U-c)\hat{\psi}' + \hat{\psi}U' = -\hat{p}, \tag{23}$$

Eliminating now $\hat{p}$ yields

$$-U'\hat{\psi}' - (U-c)\hat{\psi}'' + \hat{\psi}'U' + \hat{\psi}U'' = -k^2(U-c)\hat{\psi}. \tag{24}$$

The $U'\hat{\psi}'$ term cancels, so

$$-(U-c)\hat{\psi}'' + \hat{\psi}U'' = -k^2(U-c)\hat{\psi}. \tag{25}$$

which can also be written as

$$(U-c)\left(\partial_x^2 - k^2\right)\hat{\psi} - U''\hat{\psi} = 0, \tag{26}$$

which is known as *Rayleigh's instability equation*.

4 Rayleigh's inflection point theorem

Writing the above equation as

$$\psi'' - k^2\psi - \frac{U''}{U-c}\psi = 0 \tag{27}$$

where we have dropped the hat. Multiplying by the ψ^* and integrating over the x interval yields

$$\int \left(|\psi'| + k^2|\psi|^2\right)\,\mathrm{d}z + \int \frac{U''}{U-c}|\psi|^2 = 0. \tag{28}$$

The imaginary part of this equation is

$$\mathrm{Im}(c)\int \frac{U''}{|U-c|^2}\psi|^2 = 0. \tag{29}$$

which shows that U'' must change sign at least once in the interval.

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