Handout 8: Inflection point instability II

- Experimental studies by Reynolds (1883). Two hypotheses: viscosity acts either as to stabilize or to destabilize. No physical mechanism suggested. Unaware of earlier inviscid studies.
- Helmholtz (1868), Kelvin (1871), and Rayleigh (1880) considered the stability of inviscid incompressible flow of constant density.
- Orr (1907) and Sommerfeld (1908) considered the stability of viscous flows and confirmed that Reynolds' first hypothesis is valid.
- Fjørtoft (1950) finds a stricter necessary condition for instability. [He was part of a Princeton team that in 1950 performed the first successful numerical weather prediction using the ENIAC electronic computer.]

1 Fjørtoft's theorem

Consider again Rayleigh's equation in integral form

$$\int \left(|\psi'|^2 + k^2|\psi|^2\right)\,\mathrm{d}x + \int \frac{U''}{U-c}|\psi|^2\,\mathrm{d}x = 0, \tag{1}$$

and write $c = c_\mathrm{r} + \mathrm{i}c_\mathrm{i}$. Expand the second term with $(U-c)^* = (U-c_\mathrm{r}) - \mathrm{i}c_\mathrm{i}$. Instead of considering just the imaginary part of this equation,

$$c_\mathrm{i}\int \frac{U''}{|U-c|^2}|\psi|^2\,\mathrm{d}x = 0, \tag{2}$$

which shows that U'' must change sign at least once in the interval, he considered the real part,

$$\int \frac{U''(U-c_\mathrm{r})}{|U-c|^2}|\psi|^2\,\mathrm{d}x = -\int \left(|\psi'|^2 + k^2|\psi|^2\right)\,\mathrm{d}x < 0. \tag{3}$$

Given that

$$\int \frac{U''}{|U-c|^2}|\psi|^2\,\mathrm{d}x = 0 \quad \text{(for instability)}, \tag{4}$$

we also have

$$(c_\mathrm{r} - U_\mathrm{s})\int \frac{U''}{|U-c|^2}|\psi|^2\,\mathrm{d}x = 0 \quad \text{(for instability)}, \tag{5}$$

where $U_\mathrm{s} = U(x = x_\mathrm{s})$, and x_s is the inflection point where U'' is zero, i.e., $U''(x = x_\mathrm{s}) = 0$. Adding Equation (5) to Equation (3) yields

$$\int \frac{U''(U-U_\mathrm{s})}{|U-c|^2}|\psi|^2\,\mathrm{d}x = -\int \left(|\psi'|^2 + k^2|\psi|^2\right)\,\mathrm{d}x < 0 \quad \text{(for instability)}. \tag{6}$$

This shows that, for instability, not only has U'' to vanish at least once within the domain, but the product $U''(U-U_\mathrm{s})$ must be negative; see Figure 1 for two examples of which only one obeys $U''(U-U_\mathrm{s}) < 0$.

2 Adjoint problem

Rayleigh's instability equation,

$$(U-c)\left(\partial_x^2 - k^2\right)\hat{\psi} - U''\hat{\psi} = 0, \tag{7}$$

is not self-adjoint. The adjoint problem is given by

$$\left(\partial_x^2 - k^2\right)(U-c)\hat{\psi}^\dagger - U''\hat{\psi}^\dagger = 0. \tag{8}$$

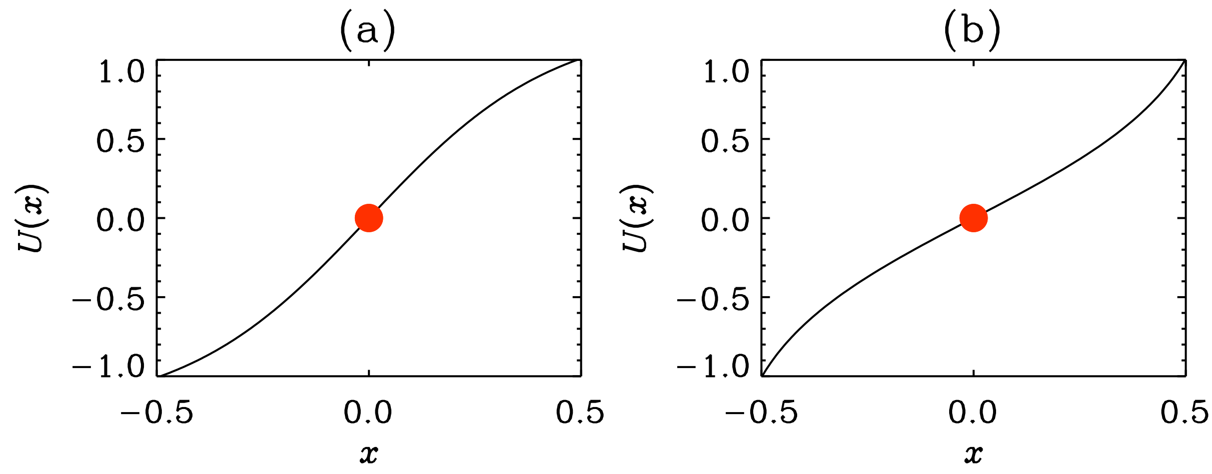

Figure 1: Sketch of two shear flow profiles. Both are Rayleigh unstable, but only one is Fjørtoft's unstable.

Differentiating through, we obtain first

$$\partial_x U' \hat{\psi}^\dagger + \partial_x (U-c)(\hat{\psi}^\dagger)' - k^2 (U-c)\hat{\psi}^\dagger - U''\hat{\psi}^\dagger = 0. \tag{9}$$

and then

$$U''\hat{\psi}^\dagger + U'(\hat{\psi}^\dagger)' + U'(\hat{\psi}^\dagger)' + (U-c)(\hat{\psi}^\dagger)'' - k^2 (U-c)\hat{\psi}^\dagger - U''\hat{\psi}^\dagger = 0. \tag{10}$$

or

$$+2U'(\hat{\psi}^\dagger)' + (U-c)(\hat{\psi}^\dagger)'' - k^2 (U-c)\hat{\psi}^\dagger = 0. \tag{11}$$

Multiplying this by $U-c$ yields

$$+2(U-c)U'(\hat{\psi}^\dagger)' + (U-c)^2(\hat{\psi}^\dagger)'' - k^2 (U-c)^2\hat{\psi}^\dagger = 0, \tag{12}$$

but since $2(U-c)U'(\hat{\psi}^\dagger)' + (U-c)^2(\hat{\psi}^\dagger)'' = \partial_x \left[(U-c)^2(\psi^\dagger)'\right]$, Equation (12) can also be written as

$$\partial_x \left[(U-c)^2 \partial_x \psi^\dagger\right] - k^2 (U-c)^2 \hat{\psi}^\dagger = 0 \tag{13}$$

which is manifestly self-adjoint!

References

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