Handout 9: Stratified shear Flows

- Stabilizing buoyancy force due to shear. Relevant in meteorology.
- Taylor, G. I. (1931) and Goldstein (1931) published first work on this in the same issue of Proc. Roy. Soc.

1 Goldstein-Taylor equation

We now swap directions and consider a shear flow of the form $U_x(z)$. In Rayleigh's instability equation, this only leads to a change of $\partial/\partial x \to \partial/\partial z$.

Rayleigh's instability equation,

$$(U-c)\left(\partial_z^2-k^2\right)\hat{\psi}-U''\hat{\psi}=0 \quad \text{(no gravity yet)}, \tag{1}$$

where primes denote now z derivatives and k denotes the x component of the wavevector; see KCD for details. Including gravity in the curled momentum equation for the y component of the vorticity $\omega_y=-\nabla^2\psi$ with stream function ψ and $\boldsymbol{u}=\boldsymbol{\nabla}\times(\psi\hat{\boldsymbol{y}})$ the term

$$\hat{\boldsymbol{y}}\cdot\boldsymbol{\nabla}\times(-g\,\rho\hat{\boldsymbol{z}})=g\,\partial\rho/\partial x. \tag{2}$$

Changes in ρ are caused by changes in the specific entropy as fluid moves up and down. Those changes are governed by

$$\frac{\mathrm{D}S}{\mathrm{D}t}=0, \tag{3}$$

Linearizing this eqation ($S=S_0(z)+S_1(x,z,t)$) about a stably stratified entropy gradient

$$\frac{\mathrm{d}S_0}{\mathrm{d}z}=N^2c_\mathrm{p}/g>0 \tag{4}$$

yields

$$\frac{\partial S_1}{\partial t}+U_x\frac{\partial S_1}{\partial x}=-N^2c_\mathrm{p}/gu_z, \tag{5}$$

where u_z is expressed in terms of ψ as $u_z=\partial\psi/\partial x$. Assuming all linearized fields to vary like $S_1=\hat{S}_1(z)\,e^{k(x-ct)}$, this yields

$$\mathrm{i}k(U-c)\,\hat{S}_1=N^2(c_\mathrm{p}/g)\mathrm{i}k\hat{\psi}, \tag{6}$$

or

$$(U-c)\,\hat{S}_1=N^2(c_\mathrm{p}/g)\hat{\psi}. \tag{7}$$

Assuming rapid pressure equilibration, we can replace entropy changes by negative density changes in Equation (2). Inserting gravity into yields

$$(U-c)\left(\partial_z^2-k^2\right)\hat{\psi}-U''\hat{\psi}+\frac{N^2}{U-c}\,\hat{\psi}=0. \tag{8}$$

Upon similar manipulations as before (details next time), one can show that

$$c_\mathrm{i}\int\frac{N^2-\frac{1}{4}(U')^2}{|U-c|^2}|\psi|^2=-c_\mathrm{i}\int\left(|\psi'|+k^2|\psi|^2\right)\,\mathrm{d}x \tag{9}$$

with $c=c_\mathrm{r}+\mathrm{i}c_\mathrm{i}$. This shows that the flow cannot be unstable if $N^2-\frac{1}{4}(U')^2>0$ everywhere in the domain. In terms of the *gradient Richardson number* $\mathrm{Ri}(z)=N^2/(U')^2$, this means $\mathrm{Ri}>1/4$ for stability.

In Figure 1, we show numerical solutions to the approximate eigenvalue problem

$$\left[U\left(\partial_z^2-k^2+N^2/U^2\right)\hat{\psi}-U''\right]\hat{\psi}=c\left(\partial_z^2-k^2-N^2/U^2\right)\hat{\psi}. \tag{10}$$

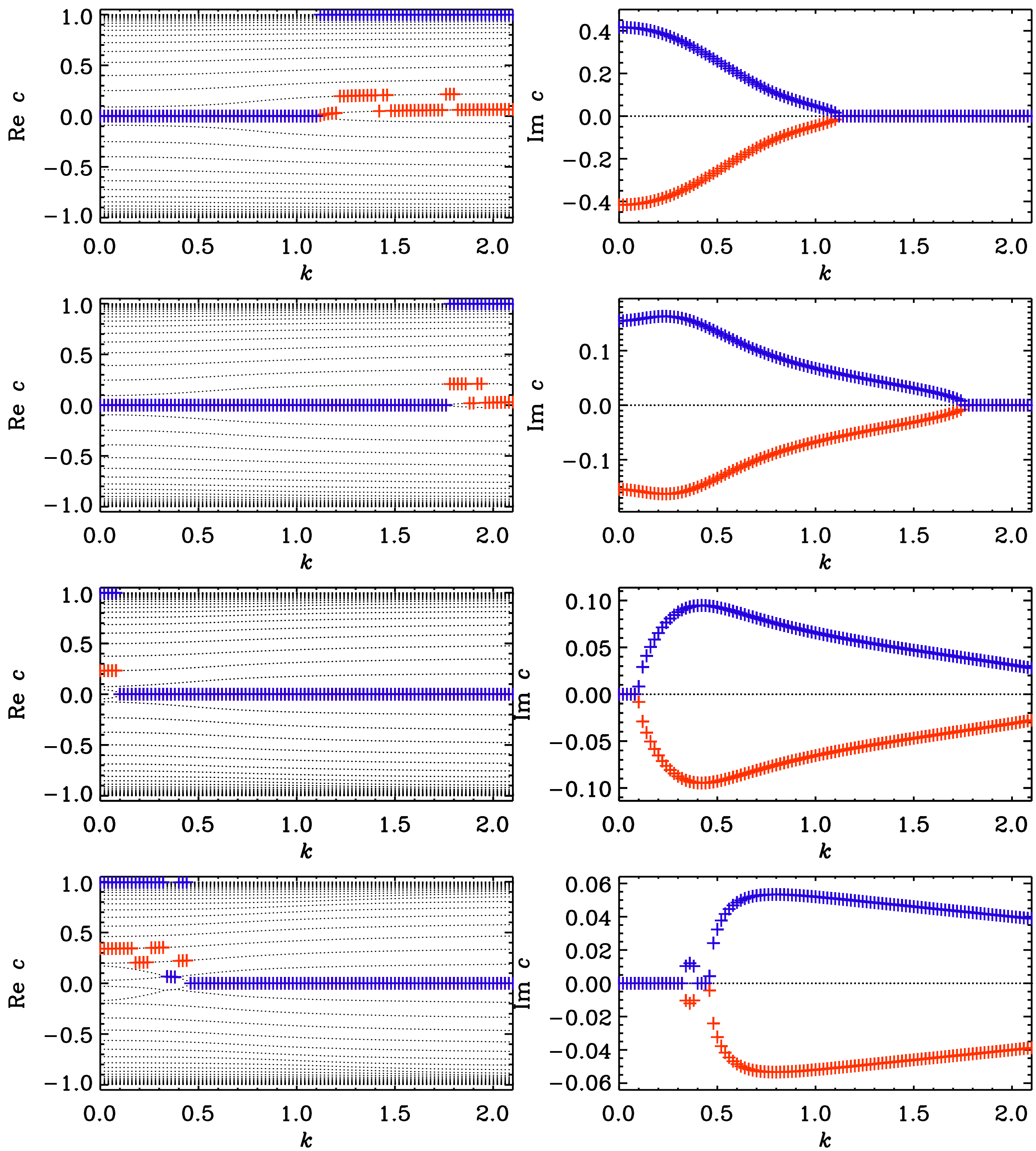

Figure 1: Eigenvalues to the approximate Taylor–Goldstein equations for $|c/U| \ll 1$ for $U = \tanh x$ in $-5 < x < 5$; see `http://lcd-www.colorado.edu/~axbr9098/teach/ASTR_5410/lectures/9_Richardson_crit/idl/`.

References

Goldstein, S., "On the stability of superposed streams of fluids of different densities," *Proc. Roy. Soc. Lond.* **132**, 524-548 (1931).

Taylor, G. I., "Effect of variation in density on the stability of superposed streams of fluid," *Proc. Roy. Soc. Lond.* **132**, 499-523 (1931).

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