PHILOSOPHICAL TRANSACTIONS.

I. *The Stability of a Spherical Nebula.*

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INTRODUCTION.

§ 1. THE object of the present paper can be best explained by referring to a sentence which occurs in a paper by Professor G. H. DARWIN.* This is as follows :—

"The principal question involved in the nebular hypothesis seems to be the stability of a rotating mass of gas; but, unfortunately, this has remained up to now an untouched field of mathematical research. We can only judge of probable results from the investigations which have been made concerning the stability of a rotating mass of liquid."

In so far as the two cases are parallel, the argument by analogy will, of course, be valid enough, but the compressibility of a gas makes possible in the gaseous nebula a whole series of vibrations which have no counterpart in a liquid, and no inference as to the stability of these motions can be drawn from an examination of the behaviour of a liquid. Thus, although there will be unstable vibrations in a rotating mass of gas similar to those which are known to exist in a rotating liquid, it does not at all follow that a rotating gas will become unstable, in the first place, through vibrations which have a counterpart in a rotating liquid: it is at any rate conceivable that the vibrations through which the gas first becomes unstable are vibrations in which the compressibility of the gas plays so prominent a part, that no vibration of the kind can occur in a liquid. If this is so, the conditions of the formation of planetary systems will be widely different in the two cases.

With a view to answering the questions suggested by this argument, the present paper attempts to examine in a direct manner the stability of a mass of gravitating gas, and it will be found that, on the whole, the results are not such as could have been predicted by analogy from the results in the case of a gravitating liquid. The

* "On the Mechanical Conditions of a Swarm of Meteorites, and on Theories of Cosmogony," 'Phil. Trans.,' A, vol. 180, p. 1 (1888).

main point of difference between the two cases can be seen, almost without mathematical analysis, as follows :—

§ 2. Speaking somewhat loosely, the stability or instability may be measured by the resultant of several factors. In the case of an incompressible liquid we may say that gravitation tends to stability, and rotation to instability; the liquid becomes unstable as soon as the second factor preponderates over the first. The gravitational tendency to stability arises in this case from the surface inequalities caused by the displacement: matter is moved from a place of higher potential to a place of lower potential, and in this way the gravitational potential energy is increased. As soon as we pass to the consideration of a compressible gas the case is entirely different.

Suppose, to take the simplest case, that we are dealing with a single shell of gravitating gas, bounded by spheres of radii r and $r + dr$, and initially in equilibrium under its own gravitation, at a uniform density ρ_0.

Suppose, now, that this gas is caused to undergo a tangential compression or dilatation, such that the density is changed from

$$\rho_0 \text{ to } \rho_0 + \Sigma \rho_n \mathrm{S}_n,$$

where ρ_n is a small quantity, and S_n is a spherical surface harmonic of order n.

It will readily be verified that there is a decrease in the gravitational energy of amount

$$4\pi r^3 (dr)^2 \Sigma \frac{\rho_n^2}{(2n+1)} \iint \mathrm{S}_n^2 \sin\theta \, d\theta \, d\phi.$$

As this is essentially a positive quantity, we see that any tangential displacement of a single shell will decrease the gravitational energy.

This example is sufficient to show that when the gas is compressible, the tendency of gravitation may be towards instability. The gravitation of the surface inequalities will as before tend towards stability, but when we are dealing with a gaseous nebula, it is impossible to suppose that a discontinuity of density can occur such as would be necessary if this tendency were to come into operation. Rotation as before will tend to instability, and the factor which makes for stability will be the elasticity of the gas.

We can now see that there is nothing inherently impossible, or even improbable, in the supposition that for a gaseous nebula the symmetrical configuration may become unstable even in the absence of rotation. The question which we shall primarily attempt to answer is, whether or not this is, in point of fact, a possible occurrence, and if so, under what circumstances it will take place. To investigate this problem, it will be sufficient to consider the vibrations of a non-rotating nebula about a configuration of spherical symmetry.

§ 3. Unfortunately, the stability of a gaseous nebula of finite size is not a subject which lends itself well to mathematical treatment. The principal difficulty lies in finding a system which shall satisfy the ordinarily assumed gas equations, and shall at the same time give an adequate representation of the primitive nebula of astronomy.

If we begin by supposing a nebula to consist of a gas which satisfies at every point the ordinarily assumed gas equations, and to be free from the influence of all external forces, then the only configuration of equilibrium is one which extends to an infinite distance, and is such that the nebula contains an infinite mass of gas. The only alternative is to suppose the gas to be totally devoid of thermal conductivity, and in this case there is an equilibrium configuration which is of finite size and involves only a finite mass of gas. But the assumption that a gas may be treated as non-conducting finds no justification in nature. When we are dealing, as in the present case, with changes extending through the course of thousands of years, we cannot suppose the gas to be such a bad conductor of heat, that any configuration, other than one of thermal equilibrium, may be regarded as permanent.

Professor DARWIN has pointed out that a nebula which consists of a swarm of meteorites may, under certain limitations, be treated as a gas of which the meteorites are the "molecules."* In this quasi-gas the mean time of describing a free path must be measured in days, rather than (as in the case of an actual gas) in units of 10^{-9} second. The process of equalisation of temperature will therefore be much slower than in the case of an actual gas, and it is possible that the conduction of heat may be so slow that it would be legitimate to regard adiabatic equilibrium as permanent.†

Except for this the mathematical conditions are identical, whether we assume the gaseous or meteoritic hypothesis. The present paper deals primarily with a nebula in which the equilibrium is conductive, but it will be found possible from the results arrived at, to obtain some insight into the behaviour of a nebula in which the equilibrium is partially or wholly convective.

§ 4. Whether we suppose the thermal equilibrium of the gas to be conductive or adiabatic, we are still met by the difficulty that the gas equations break down over the outermost part of the nebula, through the density not being sufficiently great to warrant the statistical methods of the kinetic theory. This difficulty could be avoided by supposing that the nebula is of finite size, and that equilibrium is maintained by a constant pressure applied to the outer surface of the nebula. If this pressure is so great that the density of gas at the outer surface of the nebula is sufficiently large to justify us in supposing that the gas equations are satisfied everywhere inside this surface, then the difficulty in question will have been removed. On the other hand, this pressure can only be produced in nature by the impact of matter, this matter

* G. H. DARWIN, *loc. cit., ante.*

† *Ibid.*, p. 64.

consisting either of molecules or meteorites, so that we are now called upon to take account of the gravitational forces exerted upon the nebula by this matter. This whole question is, however, deferred until a later stage; for the present we turn to the purely mathematical problem of finding the vibrations of a mass of gas which is in equilibrium in a spherical configuration. We shall consider two distinct cases. In the first, equilibrium is maintained by a constant pressure applied to the outer surface of the nebula, this surface being of radius R_1. In the second, the nebula extends to infinity, and it is assumed that the ordinary gas equations are satisfied without limitation. We suppose for the present that the gas is in thermal equilibrium throughout. It is not, however, supposed that the gas is all at the same temperature; to make the question more general, and to give a closer resemblance to the state of things which may be supposed to exist in nature, it will be supposed that the gas is collected round a solid spherical core of radius R_0, and the temperature will be supposed to fall off as we recede from this core to the surface, the equation of conduction of heat being satisfied at every point. We shall also suppose that the gas is acted upon by an external system of forces, this system being, like the nebula, spherically symmetrical. The reason for these generalisations will be seen later; it will at any time be possible to pass to less general cases.

THE CRITERION OF STABILITY.

The Principal Vibrations of a Spherical Nebula.

§ 5. We shall take the point about which the nebula is symmetrical as origin. It will be convenient to use rectangular co-ordinates x, y, z, in conjunction with polar co-ordinates r, θ, ϕ. We shall imagine the nebula to undergo a small continuous displacement; let the components of this be ξ, η, ζ, when referred to rectangular co-ordinates, and $u, rv, rw \sin\theta$ when referred to polars. Thus the point initially at

$$x, y, z \quad \text{or} \quad r, \theta, \phi$$

is found after displacement at

$$x+\xi,\ y+\eta,\ z+\zeta \quad \text{or} \quad r+u,\ \theta+v,\ \phi+w.$$

The cubical dilatation of this displacement will be denoted by Δ, so that

$$\Delta = \frac{\partial\xi}{\partial x} + \frac{\partial\eta}{\partial y} + \frac{\partial\xi}{\partial z}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(v\sin\theta) + \frac{\partial w}{\partial\phi}.$$

In general we shall denote the *density* by ρ, *pressure* by ϖ, *temperature* by T, *total potential* by V, *coefficient of conduction of heat* by κ, and the *gas constant* by λ, the last of these being given by the equation

$$\varpi = \lambda \mathrm{T} \rho \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

In the equilibrium configuration each of the quantities just defined is a function of r only.

If c is any one of these quantities, we shall denote the

Value of c in the equilibrium configuration, evaluated at x, y, z, by c_0.
,, ,, displaced ,, ,, ,, ,, $c_0 + c'$.
,, ,, ,, ,, ,, at $x+\xi, y+\eta, z+\zeta$ by $c_0 + c_1$.

The quantities c_0, c', c_1 are, of course, not independent. Since $c_0 + c_1$ is the same function of $x + \xi$, $y + \eta$, $z + \zeta$, as is $c_0 + c'$ of x, y, z, we have, as far as the first order of small quantities,

$$c_0 + c_1 = c_0 + c' + \xi \frac{\partial c_0}{\partial x} + \eta \frac{\partial c_0}{\partial y} + \zeta \frac{\partial c_0}{\partial z},$$

or, since c_0 is a function of r only,

$$c_1 = c' + u \frac{dc_0}{dr} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

§ 6. From the equation of continuity we have at once

$$\rho_1 = - \rho_0 \Delta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Since λ remains the same throughout the motion of any given element of the gas,

$$\lambda_1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Hence, from equation (1),

$$\varpi_0 + \varpi_1 = \lambda_0 (\mathrm{T}_0 + \mathrm{T}_1)(\rho_0 + \rho_1),$$

giving as the value of ϖ_1

$$\varpi_1 = \lambda_0 (\mathrm{T}_1 \rho_0 + \mathrm{T}_0 \rho_1) = \lambda_0 \rho_0 (\mathrm{T}_1 - \Delta \mathrm{T}_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

So long as we confine our attention to a single element of the gas, the coefficient of conduction of heat is proportional to the square root of the temperature, and is independent of the density.* We therefore have, as far as the first order of small quantities,

$$\frac{\kappa_1}{\kappa_0} = \frac{T_1}{2T_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

Lastly V′, regarded as the difference between $V_0 + V'$ and V_0, is seen to be the potential of a volume-distribution of matter of density ρ', to which must be added :

(i.) The potential of a surface-distribution over the sphere $r = R_0$, the surface density being

$$-\left[u(\rho_0 - \sigma_0)\right]_{r = R_0},$$

where σ_0 is the mean density of the core, and

(ii.) The potential of a surface-distribution over the sphere $r = R_1$, the surface density being

$$\left[u(\rho_0 - \sigma_1)\right]_{r = R_1},$$

where σ_1 is the density of the medium (if any) outside the nebula.

§ 7. We are now in a position to handle the equations of motion, and of conduction of heat. For the element which, in the undisturbed state, is at x, y, z, the equations of motion are three of the type

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{\partial}{\partial x}(V_0 + V') - \frac{1}{(\rho_0 + \rho')}\frac{\partial}{\partial x}(\varpi_0 + \varpi') \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

Transforming to polar co-ordinates, these equations are equivalent to

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial r}(V_0 + V') - \frac{1}{(\rho_0 + \rho')}\frac{\partial}{\partial r}(\varpi_0 + \varpi') \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

$$r\frac{\partial^2 v}{\partial t^2} = \frac{1}{r}\frac{\partial V'}{\partial \theta} - \frac{1}{\rho_0 r}\frac{\partial \varpi'}{\partial \theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

$$r \sin\theta \frac{\partial^2 w}{\partial t^2} = \frac{1}{r \sin\theta}\frac{\partial V'}{\partial \phi} - \frac{1}{\rho_0 r \sin\theta}\frac{\partial \varpi'}{\partial \phi} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

As an equation of equilibrium, we have

$$\frac{\partial V_0}{\partial r} - \frac{1}{\rho_0}\frac{\partial \varpi_0}{\partial r} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11),$$

and with the help of this, equation (8) reduces to

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial V'}{\partial r} - \frac{1}{\rho_0}\frac{\partial \varpi'}{\partial r} + \frac{\rho'}{\rho_0^2}\frac{\partial \varpi_0}{\partial r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12),$$

as far as the first order of small quantities.

* BOLTZMANN, 'Vorlesungen über Gastheorie,' vol. 1, § 13.

Let us write

$$\chi = \mathrm{V}' - \varpi'/\rho_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13),$$

so that

$$\frac{\partial \chi}{\partial r} = \frac{\partial \mathrm{V}'}{\partial r} - \frac{1}{\rho_0}\frac{\partial \varpi'}{\partial r} + \frac{\varpi'}{\rho_0^2}\frac{\partial \rho_0}{\partial r},$$

then equation (12) becomes

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial \chi}{\partial r} + \frac{1}{\rho_0^2}\left(\rho'\frac{\partial \varpi_0}{\partial r} - \varpi'\frac{\partial \rho_0}{\partial r}\right),$$

and, by the use of equation (2), this is seen to be equivalent to

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial \chi}{\partial r} + \frac{1}{\rho_0^2}\left(\rho_1\frac{\partial \varpi_0}{\partial r} - \varpi_1\frac{\partial \rho_0}{\partial r}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (14).$$

Equations (9) and (10) now take the simple forms,

$$\frac{\partial^2 v}{\partial t^2} = \frac{1}{r^2}\frac{\partial \chi}{\partial \theta} \qquad \frac{\partial^2 w}{\partial t^2} = \frac{1}{r^2 \sin^2\theta}\frac{\partial \chi}{\partial \phi}.$$

From these last two equations, we obtain at once

$$\frac{\partial^2}{\partial t^2}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(v\sin\theta) + \frac{\partial w}{\partial\phi}\right\} = \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\chi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\chi}{\partial\phi^2},$$

or, what is the same thing,

$$\frac{\partial^2}{\partial t^2}\left\{\Delta - \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u)\right\} = \nabla^2\chi - \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\chi}{\partial r}\right) \quad . \quad . \quad . \quad . \quad . \quad (15).$$

§ 8. The equation of conduction of heat is, as far as the first order of small quantities,

$$-\mathrm{M}\frac{\partial\rho_1}{\partial t} + \mathrm{C}_v\frac{\partial \mathrm{T}_1}{\partial t} = \frac{1}{\rho}\left\{\frac{\partial}{\partial x}\left(\kappa\frac{\partial \mathrm{T}}{\partial x}\right) + \frac{\partial}{\partial y}\left(\kappa\frac{\partial \mathrm{T}}{\partial y}\right) + \frac{\partial}{\partial z}\left(\kappa\frac{\partial \mathrm{T}}{\partial z}\right)\right\} . \quad . \quad . \quad (16),$$

in which ρ, κ, T stand for $\rho_0 + \rho'$, $\kappa_0 + \kappa'$, $\tau_0 + \tau'$ respectively. The notation is that of KIRCHHOFF; the equation may either be written down from first principles, or regarded as a simplified form of KIRCHHOFF's general equation.*

Since there is thermal equilibrium in the undisturbed configuration,

$$\frac{\partial}{\partial x}\left(\kappa_0\frac{\partial \mathrm{T}_0}{\partial x}\right) + \frac{\partial}{\partial y}\left(\kappa_0\frac{\partial \mathrm{T}_0}{\partial y}\right) + \frac{\partial}{\partial z}\left(\kappa_0\frac{\partial \mathrm{T}_0}{\partial z}\right) = 0. \quad . \quad . \quad . \quad . \quad . \quad (17).$$

* KIRCHHOFF, 'Vorlesungen über die Theorie der Wärme, p. 118.

Hence equation (16) reduces to the form

$$-\mathrm{M}\frac{\partial\rho_1}{\partial t}+\mathrm{C}_v\frac{\partial\mathrm{T}_1}{\partial t}=\frac{1}{\rho_0}\left\{\frac{\partial}{\partial x}\left(\kappa_0\frac{\partial\mathrm{T}'}{\partial x}\right)+\frac{\partial}{\partial y}\left(\kappa_0\frac{\partial\mathrm{T}'}{\partial y}\right)+\frac{\partial}{\partial z}\left(\kappa_0\frac{\partial\mathrm{T}'}{\partial z}\right)\right.$$
$$\left.+\frac{\partial}{\partial x}\left(\kappa'\frac{\partial\mathrm{T}_0}{\partial x}\right)+\frac{\partial}{\partial y}\left(\kappa'\frac{\partial\mathrm{T}_0}{\partial y}\right)+\frac{\partial}{\partial z}\left(\kappa'\frac{\partial\mathrm{T}_0}{\partial z}\right)\right\} \quad . \quad (18).$$

Since κ_0, T_0 are functions of r only, the bracket on the right-hand side of this last equation again reduces to

$$\frac{\partial\kappa_0}{\partial r}\frac{\partial\mathrm{T}'}{\partial r}+\kappa_0\nabla^2\mathrm{T}'+\frac{\partial\kappa'}{\partial r}\frac{\partial\mathrm{T}_0}{\partial r}+\kappa'\nabla^2\mathrm{T}_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19),$$

and, cleared of accented symbols by the use of equation (2), this takes the form

$$\frac{\partial\kappa_0}{\partial r}\frac{\partial\mathrm{T}_1}{\partial r}+\kappa_0\nabla^2\mathrm{T}_1+\frac{\partial\kappa_1}{\partial r}\frac{\partial\mathrm{T}_0}{\partial r}+\kappa_1\nabla^2\mathrm{T}_0$$
$$-u\left\{\frac{\partial}{\partial r}\left(\frac{\partial\kappa_0}{\partial r}\frac{\partial\mathrm{T}_0}{\partial r}\right)+\kappa_0\nabla^2\left(\frac{\partial\mathrm{T}_0}{\partial r}\right)+\frac{\partial\kappa_0}{\partial r}\nabla^2\mathrm{T}_0\right\}$$
$$-2\frac{\partial u}{\partial r}\left\{\frac{\partial\kappa_0}{\partial r}\frac{\partial\mathrm{T}_0}{\partial r}+\kappa_0\frac{\partial^2\mathrm{T}_0}{\partial r^2}\right\}-\kappa_0\frac{\partial\mathrm{T}_0}{\partial r}\nabla^2 u. \quad . \quad . \quad . \quad . \quad (20).$$

Now equation (17) can be written in the form

$$\frac{\partial\kappa_0}{\partial r}\frac{\partial\mathrm{T}_0}{\partial r}+\kappa_0\nabla^2\mathrm{T}_0=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21),$$

whence, by differentiation with respect to r,

$$\frac{\partial}{\partial r}\left(\frac{\partial\kappa_0}{\partial r}\frac{\partial\mathrm{T}_0}{\partial r}\right)+\kappa_0\frac{\partial}{\partial r}\nabla^2\mathrm{T}_0+\frac{\partial\kappa_0}{\partial r}\nabla^2\mathrm{T}_0=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (22).$$

With the help of equation (22), the bracket in the second line of (20) reduces to

$$\frac{2\kappa_0}{r^2}\frac{\partial\mathrm{T}_0}{\partial r},$$

while, with the help of (21), that in the third line becomes

$$-\frac{2\kappa_0}{r}\frac{\partial\mathrm{T}_0}{\partial r}.$$

Again, if we substitute for κ_1 the value found for it in equation (6), the two last terms in the first line of (20) can be transformed as follows:

$$\frac{\partial\kappa_1}{\partial r}\frac{\partial\mathrm{T}_0}{\partial r}+\kappa_1\nabla^2\mathrm{T}_0=\kappa_0\frac{\partial}{\partial r}\left(\frac{\mathrm{T}_1}{2\mathrm{T}_0}\right)\frac{\partial\mathrm{T}_0}{\partial r}+\frac{\mathrm{T}_1}{2\mathrm{T}_0}\left\{\frac{\partial\kappa_0}{\partial r}\frac{\partial\mathrm{T}_0}{\partial r}+\kappa_0\nabla^2\mathrm{T}_0\right\},$$

and the last bracket vanishes by equation (21).

Collecting results, and substituting for ρ_1 from equation (3), we find that equation (18) takes the form

$$M\rho_0 \frac{\partial \Delta}{\partial t} + C_v \frac{\partial T_1}{\partial t} = \frac{1}{\rho}\left\{\frac{\partial \kappa_0}{\partial r}\frac{\partial T_1}{\partial r} + \kappa_0 \nabla^2 T_1 + \kappa_0 \frac{\partial}{\partial r}\left(\frac{T_1}{2T_0}\right)\frac{\partial T_0}{\partial r} - \kappa_0 \frac{\partial T_0}{\partial r}\left(\frac{2u}{r^2} - \frac{4}{r}\frac{\partial u}{\partial r} + \nabla^2 u\right)\right\} \quad . \quad . \quad . \quad (23).*$$

§ 9. In addition to the volume-equations which have just been found, there are certain boundary conditions which must be satisfied. These are as follows:

(i.) The pressure must remain constant at the outer surface, so that we must have

$$[\varpi_1]_{r=R_1} = 0.$$

(ii.) The temperature must remain unaltered at $r = R_0$, or else the flow of temperature across the surface $r = R_0$ must remain *nil*. These two suppositions require respectively

$$[T_1]_{r=R_0} = 0, \quad \text{or} \quad \left[\frac{dT_1}{dr}\right]_{r=R_0} = 0.$$

(iii.) A similar temperature condition must be satisfied at $r = R_1$.

(iv.) The kinematical and dynamical boundary conditions at the surface $r = R_0$ must be satisfied. These express that the normal velocities shall be continuous at this surface, and that the motion of the rigid core shall be such as would be caused by the forces acting upon it from the gas.

§ 10. Equations (14), (15) and (23) give the rates of change in u, Δ and T_1 in terms of these quantities. Hence these equations enable us theoretically to trace the changes in u, Δ and T_1, starting from any arbitrary values of u, Δ, T_1, du/dt and $d\Delta/dt$, which are such as to satisfy the boundary conditions.

Imagine initial values of u, Δ, T_1, du/dt and $d\Delta/dt$, in which the latitude and longitude enter only through the factor S_n, where S_n is any spherical harmonic of order n. Then it can be shown that the solution through all time (so long as the squares of the displacement may be neglected) is such that the latitude and longitude enter only through the factor S_n. For, assuming a solution of this form, the value of V′ found in § 6 will contain S_n as a factor, as will also ρ_1, ϖ_1, ϖ' (equations 3, 5, 2) and χ (equation 13). The same is true of $\nabla^2\chi$, $\nabla^2 T_1$ and $\nabla^2 u$, since

$$\nabla^2 [f(r) S_n] = \left\{\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{df(r)}{dr}\right) - \frac{n(n+1)f(r)}{r^2}\right\} S_n,$$

* Sections 5–8 were re-written in November, 1901. I take this opportunity of expressing my thanks to the referee for the care and trouble which he has bestowed upon my paper. To him I am indebted for several improvements in these four sections, in particular for the present form of equation (23), and also for the removal of a serious inaccuracy from my original equations.

where $f(r)$ is any function of r. It therefore appears that every term in equations (14), (15) and (23) will contain S_n as a factor. Dividing out by this factor, we are left with equations which do not involve θ and ϕ, and this verifies our statement.

§ 11. It therefore follows that there are principal vibrations* in which u, Δ and T_1 are of the form

$$u = AS_n e^{ipt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24),$$

$$\Delta = BS_n e^{ipt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25),$$

$$T_1 = CS_n e^{ipt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26),$$

in which A, B, C are functions of r only. The relations between A, B, C and p must be found from the equations (14), (15), (23), and the boundary conditions.

The value of ρ' for the vibration just specified is

$$\rho' = \rho_1 - u\frac{d\rho_0}{dr} = -\left(\Delta\rho_0 + u\frac{d\rho_0}{dr}\right) = -\left(A\frac{d\rho_0}{dr} + B\rho_0\right)S_n e^{ipt}.$$

We shall in future drop all zero suffixes, there being no longer any danger of confusion. Calculating V′ after the manner explained in § 6, we find (*cf.* THOMSON and TAIT, 'Nat. Phil.,' § 542),

$$V' = VS_n e^{ipt},$$

where

$$V = \frac{4\pi}{(2n+1)\,r^{n+1}}\left\{-\int_{R_0}^{r}\left(A\frac{d\rho}{dr} + B\rho\right)r^{n+2}\,dr - \left[A\,(\rho - \sigma_0)\,r^{n+2}\right]_{r=R_0}\right\}$$
$$+ \frac{4\pi r^n}{(2n+1)}\left\{-\int_{r}^{R_1}\left(A\frac{d\rho}{dr} + B\rho\right)\frac{dr}{r^{n-1}} + \left[\frac{A\,(\rho - \sigma_1)}{r^{n-1}}\right]_{r=R_1}\right\} \quad . \quad . \quad . \quad (27).$$

We have further, by equations (2) and (5),

$$\varpi' = \varpi_1 - u\frac{d\varpi}{dr} = \lambda\rho\,(C - BT)\,S_n e^{ipt} - A\frac{d\varpi}{dr}S_n e^{ipt},$$

and hence we obtain (equation 13)

$$\chi = FS_n e^{ipt},$$

where

$$F = V - \lambda\,(C - BT) + \frac{A}{\rho}\frac{d\varpi}{dr} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28).$$

Substituting the assumed solutions for u, Δ and T_1, and the corresponding values for χ, ρ_1, ϖ_1, in equations (14), (15) and (23), and dividing throughout by the factor $S_n e^{ipt}$, we find the relations

* In order to avoid circumlocution, we shall find it convenient to use the terms "principal co-ordinate" and "principal vibration," although we are ignorant as to whether the nebula is stable or unstable. It will ultimately be found that we only apply our results to nebulæ which are either stable or in the limiting state of neutral equilibrium.

$$-p^2 A = \frac{dF}{dr} - \frac{B}{\rho}\frac{d\varpi}{dr} - \frac{\lambda}{\rho}\frac{d\rho}{dr}(C - BT) \quad . \quad . \quad . \quad . \quad . \quad (29),$$

$$-p^2\left(B - \frac{1}{r^2}\frac{d}{dr}(r^2 A)\right) = -\frac{n(n+1)}{r^2}F \quad . \quad . \quad . \quad . \quad . \quad (30),$$

$$ip\rho(M\rho B + C_v C) = \frac{d\kappa}{dr}\frac{dC}{dr} + \frac{\kappa}{r^2}\left\{\frac{d}{dr}\left(r^2\frac{dC}{dr}\right) - n(n+1)C\right\} + \kappa\frac{d}{dr}\left(\frac{C}{2T}\right)\frac{dT}{dr}$$
$$- \kappa\frac{dT}{dr}\left\{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dA}{dr}\right) - \frac{4}{r}\frac{dA}{dr} - \frac{n(n+1)-2}{r^2}A\right\} \quad . \quad (31).$$

The boundary-equations found in § 9 reduce to the following :—

(i) $[C - BT]_{r=R_1} = 0$ . . . . (32),

(ii) $C_{r=R_0} = 0$ or $\left[\frac{dC}{dr}\right]_{r=R_0} = 0$. . . . . (33),

(iii) Equations similar to (33) at $r = R_1$ . . . . (34),

(iv) $(A)_{r=R_0} = 0$, when n is different from unity, or a more complex equation in the case of $n = 1$ . . . . (35).

§ 12. From the manner in which the analysis has been conducted, it will be clear that every principal vibration must either be one of the class just investigated, or else a vibration such that u, Δ, and T vanish everywhere.

For the latter class of vibration there are no forces of restitution. Thus the frequency of vibration is zero, and the motion consists of the flow of the gas in closed circuits, this flow being entirely tangential, and the gas behaving like an incompressible fluid. Obviously these steady currents are of no importance in connection with the question of stability or instability.

Discussion of the Frequency Equation.

§ 13. Returning to the class of vibrations in which u, Δ, and T do not all vanish, we have seen that the frequency equation is found by the elimination of F, A, B, and C from equations (28) to (35). Now p only enters into three of these equations; namely (31), in which it enters through the factor ip, and (29) and (30), in which it enters through the factor $-p^2$ or $(ip)^2$. Regarding ip, A, B, C, and F as unknowns, it will be seen that the coefficients which occur in equations (28) to (35) are all real. The four volume equations enable us to determine A, B, C, and F except for certain constants of integration, and the values of these quantities will be wholly real if ip is real. The boundary-equations enable us to determine the constants of integration and also provide an equation for ip. Every term in these equations will be real if ip is real. Hence the frequency equation can be written in the form

$$f(ip) = 0,$$

where $f(x)$ is a function of x in which all the coefficients are real, these coefficients being functions of n and of the quantities which determine the equilibrium configuration of the nebula.

It follows that the complex roots of ip will occur in pairs of the form

$$ip = \gamma \pm i\,\delta,$$

where γ and δ are both real. There may also be roots for which ip is purely real, so that $\delta = 0$, and γ exists alone.

The vibration corresponding to any root is stable or unstable according as γ is negative or positive.

If the equilibrium configuration of the nebula changes in any continuous manner, so as always to remain an equilibrium configuration, the values of ip will also change in a continuous manner, and for physical reasons these values can never become infinite. Hence, if the configuration of the nebula changes from one of stability to one of instability, it must do so by passing through a configuration in which there is a vibration for which $\gamma = 0$.

§ 14. For the present we shall not discuss the actual stability or instability of any configuration, but shall examine under what circumstances a transition from stability to instability can occur.

We therefore proceed to search for configurations in which there are vibrations such that $\gamma = 0$. Now for such a vibration we have either a root of the frequency equation $p = 0$, or else a pair of roots of the form $ip = \pm\, i\,\delta$.

In the latter case the corresponding vibration is one in which a dissipation of energy does not occur. A necessary condition for such a vibration is that no conduction of heat shall take place. Hence both sides of the equation of conduction of heat (equation 31) must vanish. Excluding adiabatic motion (represented by the vanishing of the factor $M\rho B + C_v C$), this condition compels us to take

$$p = 0$$

together with

$$\frac{d\kappa}{dr}\frac{dC}{dr} + \frac{\kappa}{r^2}\left\{\frac{d}{dr}\left(r^2\frac{dC}{dr}\right) - n(n+1)C\right\} + \kappa\frac{d}{dr}\left(\frac{C}{2T}\right)\frac{dT}{dr}$$

$$- \kappa\frac{dT}{dr}\left\{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dA}{dr}\right) - \frac{4}{r}\frac{dA}{dr} - \frac{n(n+1)-2}{r^2}A\right\} = 0 \quad . \quad . \quad (36).$$

Thus vibrations for which $\gamma = 0$, if they exist, must satisfy equations (32) to (36), and also equations (29) and (30), in which p is put equal to zero, and equation (28).

The case of $n = 0$ will be considered later (§ 28). Excluding this for the present, we find that putting $p = 0$ in (30) leads to

$$\mathrm{F} = 0 \qquad (37).$$

Equation (29) now reduces to

$$\mathrm{B}\frac{d\varpi}{dr} + \lambda\frac{d\rho}{dr}(\mathrm{C} - \mathrm{BT}) = 0 \qquad (38),$$

or, replacing ϖ by its value $\lambda\mathrm{T}\rho$,

$$\mathrm{B}\rho\frac{d}{dr}(\lambda\mathrm{T}) + \lambda\frac{d\rho}{dr}\mathrm{C} = 0 \qquad (39).$$

Equation (28) becomes

$$\mathrm{V} = \lambda(\mathrm{C} - \mathrm{BT}) - \frac{\mathrm{A}}{\rho}\frac{d\varpi}{dr} = 0 \qquad (40),$$

and the elimination of $\mathrm{C} - \mathrm{BT}$ from this equation and (38) leads to the equation

$$\frac{d\rho}{dr}\mathrm{V} = -\left(\mathrm{A}\frac{d\rho}{dr} + \mathrm{B}\rho\right)\frac{1}{\rho}\frac{d\varpi}{dr} \qquad (41).$$

Substituting for V from equation (27), this becomes

$$\frac{4\pi}{(2n+1)r^{n+1}}\left\{\int_{\mathrm{R}_0}^{r}\left(\mathrm{A}\frac{d\rho}{dr} + \mathrm{B}\rho\right)r^{n+2}\,dr + \left[\mathrm{A}(\rho - \sigma_0)\,r^{n+2}\right]_{r=\mathrm{R}_0}\right\}$$
$$+ \frac{4\pi r^n}{(2n+1)}\left\{\int_{r}^{\mathrm{R}_1}\left(\mathrm{A}\frac{d\rho}{dr} + \mathrm{B}\rho\right)\frac{dr}{r^{n-1}} - \left[\frac{\mathrm{A}(\rho - \sigma_1)}{r^{n-1}}\right]_{r=\mathrm{R}_1}\right\}$$
$$= \left(\mathrm{A}\frac{d\rho}{dr} + \mathrm{B}\rho\right)\frac{1}{\rho}\frac{d\varpi}{dr}\Big/\frac{d\rho}{dr} \qquad (42).$$

§ 15. With a view to transforming this equation, let us consider the equation

$$\frac{4\pi}{(2n+1)r^{n+1}}\left\{\int_{\mathrm{R}_0}^{r}\mathrm{J}r^{n+2}\,dr + \mathrm{K}_0\right\} + \frac{4\pi r^n}{(2n+1)}\left\{\int_{r}^{\mathrm{R}_1}\frac{\mathrm{J}}{r^{n-1}}\,dr + \mathrm{K}_1\right\} = \mathrm{L} \quad (43).$$

in which J and L are any functions of r, and K_0, K_1 are constants. If we multiply by r^{n+1}, and differentiate with respect to r, we obtain, after some simplification,

$$4\pi r^{2n}\left\{\int_{r}^{\mathrm{R}_1}\frac{\mathrm{J}}{r^{n-1}}\,dr + \mathrm{K}_1\right\} = \frac{d}{dr}(\mathrm{L}r^{n+1}) \qquad (44),$$

while by multiplying (43) by r^{-n} and differentiating, we obtain in a similar way

$$-\frac{4\pi}{r^{2n+2}}\left\{\int_{\mathrm{R}_0}^{r}\mathrm{J}r^{n+2}\,dr + \mathrm{K}_0\right\} = \frac{d}{dr}(\mathrm{L}r^{-n}) \qquad (45).$$

Divide (44) by r^{2n} and differentiate with respect to r, then

$$-\frac{4\pi J}{r^{n-1}} = \frac{d}{dr}\left\{\frac{1}{r^{2n}}\frac{d}{dr}\left(Lr^{n+1}\right)\right\} \quad . \quad . \quad . \quad . \quad (46),$$

or, writing ξ for Lr, and simplifying

$$\frac{d^2\xi}{dr^2} - \frac{n(n+1)}{r^2}\xi = -4\pi r J \quad . \quad . \quad . \quad . \quad (47),$$

and this same equation could have been deduced from (45) instead of (44).

Equation (47) is more general than (43) since the two constants K_0, K_1 have disappeared. In fact equation (47), being a differential equation of the second order, will contain two arbitrary constants in its solution, and these correspond to the two missing constants K_0 and K_1. We can, however, determine K_0, K_1 in terms of these two arbitrary constants, and if these constants are chosen so as to give the right values for K_0, K_1, the solution of (47) will be equivalent to the original equation (43).

To determine K_0, K_1, put $r = R_1$ in (44) and we obtain

$$4\pi K_1 = \left[\frac{1}{r^{2n}}\frac{d}{dr}\left(\xi r^n\right)\right]_{r=R_1} \quad . \quad . \quad . \quad . \quad (48),$$

and similarly from (45)

$$4\pi K_0 = -\left[r^{2n+2}\frac{d}{dr}\left(\xi r^{-(n+1)}\right)\right]_{r=R_0} \quad . \quad . \quad . \quad . \quad (49).$$

Hence we see that equation (43) is exactly equivalent to the three equations (47), (48), and (49).

§ 16. Comparing (42) with (43), it appears that (42) is exactly equivalent to the following equations :—

$$\xi = \left(A\frac{d\rho}{dr} + B\rho\right)\frac{r}{\rho}\frac{d\varpi}{dr}\Big/\frac{d\rho}{dr} \quad . \quad . \quad . \quad . \quad (50).$$

$$\frac{d^2\xi}{dr^2} - \frac{n(n+1)}{r^2}\xi = -4\pi r\left(A\frac{d\rho}{dr} + B\rho\right) \quad . \quad . \quad . \quad (51).$$

$$4\pi K_1 \equiv \left[\frac{1}{r^{2n}}\frac{d}{dr}\left(\xi r^n\right)\right]_{r=R_1} = -\left[\frac{A(\rho - \sigma_1)}{r^{n-1}}\right]_{r=R_1} \quad . \quad . \quad . \quad (52).$$

$$4\pi K_0 \equiv -\left[r^{2n+2}\frac{d}{dr}\left(\xi r^{-(n+1)}\right)\right]_{r=R_0} = \left[A(\rho - \sigma_0)\, r^{n+2}\right]_{r=R_0} \quad . \quad . \quad . \quad (53).$$

The right-hand member of (51) is equal to

$$-4\pi\xi\frac{d\rho}{dr}\Big/\frac{1}{\rho}\frac{d\varpi}{dr},$$

so that if we introduce a new quantity u, defined by

$$u = 2\pi\rho r^2 \frac{d\rho}{dr} \Big/ \frac{d\varpi}{dr} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54),$$

equation (51) may be written in the form

$$r^2 \frac{d^2\xi}{dr^2} = \{n(n+1) - 2u\}\,\xi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55).$$

The solution of this will be of the form

$$\xi = E_1\phi_1(r) + E_2\phi_2(r) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56),$$

in which E_1, E_2 are constants of integration. We have, from the definition of ξ,

$$A\frac{d\rho}{dr} + B\rho = \frac{\xi u}{2\pi r^3} = \frac{u}{2\pi r^3}\{E_1\phi_1(r) + E_2\phi_2(r)\} \quad . \quad . \quad . \quad . \quad (57),$$

and the elimination of B from this equation and (39) gives

$$\lambda\frac{d\rho}{dr}\,C = \frac{d}{dr}(\lambda T)\left\{A\frac{d\rho}{dr} - \frac{u}{2\pi r^3}\{E_1\phi_1(r) + E_2\phi_2(r)\}\right\} \quad . \quad . \quad . \quad (58).$$

If we imagine this value for C substituted in equation (36), we shall have a differential equation of the second order for A. The solution of this will be of the form

$$A = E_1 f_1(r) + E_2 f_2(r) + E_3 f_3(r) + E_4 f_4(r) \quad . \quad . \quad . \quad . \quad (59),$$

in which E_3 and E_4 are the new constants of integration. From this value of A we can deduce the values of B and C (equations (57) and (58)) without introducing any further constants of integration.

Turning to the boundary conditions, we now find that there are six boundary-equations to be satisfied (equations (32), (33), (34), (35), (52), (53)) and only three arbitrary constants at our disposal, namely, the ratios of the four E's. If we eliminate these E's we shall be left with three equations to determine the configuration of the nebula at which instability sets in, and these equations will in general be inconsistent.

§ 17. In order to put the right interpretation upon this result, it will be necessary to return to the general equations of free vibrations found in § 12.

If we eliminate F from equations (29) and (30), we obtain

$$p^2 \left\{ \mathrm{A} + \frac{1}{n(n+1)} \frac{d}{dr}(r^2\mathrm{B}) - \frac{d}{dr}(r^2\mathrm{A}) \right\} = \frac{\lambda}{\rho} \frac{d\rho}{dr} \mathrm{C} + \frac{d}{dr}(\lambda\mathrm{T})\,\mathrm{B} \quad . \quad . \quad (60),$$

while equation (30) may, with the help of (28), be written in the form

$$\mathrm{V} = \xi / r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61),$$

in which ξ is now defined by

$$\frac{\xi}{r} = \lambda(\mathrm{C} - \mathrm{BT}) - \frac{\mathrm{A}}{\rho} \frac{d\varpi}{dr} + \frac{p^2}{n(n+1)} \left\{ r^2\mathrm{B} - \frac{d}{dr}(r^2\mathrm{A}) \right\} \quad . \quad . \quad . \quad (62).$$

Substituting for V from equation (27), and treating the equation so formed in the manner explained in § 15, we find, as the equivalent of equation (61),

(i.) A volume equation, analogous in form to (51), namely,

$$\frac{d^2\xi}{dr^2} - \frac{n(n+1)}{r^2}\xi = -4\pi r\left(\mathrm{A}\frac{d\rho}{dr} + \mathrm{B}\rho\right) \quad . \quad . \quad . \quad . \quad . \quad (63).$$

(ii.) Two boundary equations analogous in form to (52) and (53). . . (64), (65).

Thus the equations found in § 11 may be replaced by

(α) Three volume equations, namely, equations (60), (63), and (31).

(β) Six boundary equations, namely, equations (32), (33), (34), (35), (64), (65).

We may conduct the elimination of B and C from the three equations (α) in a symbolic manner as follows:—

Let D_n be a symbol which is used to denote any linear differential operator of order n, the differentiations being with respect to r. The symbol has reference solely to the order of the highest differential coefficient which occurs, and must in no case have reference to any particular differential operator. Thus we write D_n indiscriminately for every operator of the form

$$f_n(r)\frac{\partial^n}{\partial r^n} + f_{n-1}(r)\frac{\partial^{n-1}}{\partial r^{n-1}} + \ldots$$

The laws governing the manipulation of this symbol are as follows:

$$\begin{aligned} &\text{(i.)} & -\mathrm{D}_n\phi &= \mathrm{D}_n\phi, \\ &\text{(ii.)} & \mathrm{D}_n\phi + \mathrm{D}_m\phi &= \mathrm{D}_n\phi \ (n > m), \\ &\text{(iii.)} & \mathrm{D}_n(\mathrm{D}_m\phi) &= \mathrm{D}_{m+n}\phi. \end{aligned}$$

It must be particularly noticed that in general

$$D_n\phi - D_n\phi = D_n\phi.$$

Corresponding, however, to any two specified operators of order n, say $(D_n)_1$ and $(D_n)_2$, it will always be possible to find two functions of r, say a and b, such that

$$a(D_n)_1\phi - b(D_n)_2\phi = D_{n-1}\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66).$$

In terms of this operator, the three equations (α) (p. 16) may be written in the following forms:

$$p^2(D_2A + D_1B) + D_0B + D_0C = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (67),$$

$$D_2A + D_2B + D_2C + p^2(D_3A + D_2B) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (68),$$

$$ip(D_0B + D_0C) + D_2C + D_2A = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69).$$

Now D_n is commutative with regard to functions of r, and is of course commutative with regard to p. This enables us to eliminate B and C from the above equations.

To make this clearer, consider a simple case, say the pair of equations

$$D_2A = D_nB \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70).$$

$$D_1A = p^2D_mB \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71).$$

If we operate on (71) with d/dr, we get an equation of the form

$$D_2A = p^2D_{m+1}B,$$

and from this and equation (70), we can, with the help of the property expressed in equation (66), deduce an equation of the form

$$D_1A = D_nB + p^2D_{m+1}B.$$

From this and equation (71) we can in a similar way obtain an equation of the form

$$D_0A = D_nB + p^2D_{m+1}B.$$

We may regard this as an equation giving A, and substitute for A in (71). In this way we obtain

$$D_{n+1}B + p^2D_{m+2}B = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72),$$

and the elimination of A has been effected.

It will be clear that throughout this elimination we have followed a method which would have been successful in eliminating A if d/dr had been regarded as a mere multiplier. The result of the elimination is accordingly exactly the same as might have been obtained directly from the original equations (70) and (71), by regarding the D's as multipliers and eliminating according to the ordinary laws of algebra, It will now be apparent that we can eliminate any two of the three unknowns, A, B, and C, from equations (67)–(69) by this method. The differential equation satisfied by the remaining unknown (say A) will be

$$\Delta \mathrm{A} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73),$$

where, symbolically,

$$\Delta \equiv \begin{vmatrix} p^2 \mathrm{D}_2, & p^2 \mathrm{D}_1 + \mathrm{D}_0, & \mathrm{D}_0 \\ p^2 \mathrm{D}_3 + \mathrm{D}_2, & p^2 \mathrm{D}_2 + \mathrm{D}_2, & \mathrm{D}_2 \\ \mathrm{D}_2, & ip\mathrm{D}_0, & ip\mathrm{D}_0 + \mathrm{D}_2 \end{vmatrix} \quad . \quad . \quad . \quad . \quad (74).$$

We may expand this determinant according to the rules already laid down for the manipulation of the D's, and so obtain

$$\Delta = ip^5 \mathrm{D}_4 + p^4 \mathrm{D}_6 + ip^3 \mathrm{D}_4 + p^2 \mathrm{D}_6 + ip\mathrm{D}_2 + \mathrm{D}_4 \quad . \quad . \quad . \quad . \quad (75).$$

§ 18. We can now see the explanation of the difficulty which occurred in § 16. The occurrence of the term D_6 in Δ points to a differential equation of the sixth order, which is satisfied by any one of the quantities A, B, or C in the general case, in which p does not vanish. As soon, however, as p is put equal to zero, the expression for Δ reduces to D_4, and the differential equation is one of the fourth order only. It therefore appears that by putting $p = 0$ *before* solving the differential equations, the order of these equations is reduced automatically, and two solutions are entirely lost from sight.

These two last solutions, it is easy to see, are solutions which do not approximate to a definite limit, when p approximates to zero. The remaining four solutions will approximate to the same forms as would be obtained by putting $p = 0$ before solving the differential equations. Thus, instead of equation (59), we must write the complete limiting solution for A in the form

$$\mathop{\mathrm{Lt}}_{p=0} \mathrm{A} = \mathrm{E}_1 f_1(r) + \mathrm{E}_2 f_2(r) + \mathrm{E}_3 f_3(r) + \mathrm{E}_4 f_4(r) + \mathrm{E}_5 f_5(r, p) + \mathrm{E}_6 f_6(r, p)^* \; . \; (76).$$

* I have not found it possible to investigate the form of these two last solutions in the general case, but it is easy to examine the nature of the solutions at infinity, when the nebula extends to infinity, and this enables us to form some idea as to the general nature of the solutions. Suppose that at infinity we have

$$\mathop{\mathrm{Lt}}_{r=\infty} \frac{1}{\lambda\rho} \frac{d\varpi}{dr} \frac{d\lambda}{dr} = \pm a^2 r^{-s},$$

in which a is real, then it can be shown that $\mathrm{A} = \phi(r, p)\, \mathrm{A}'$, &c., in which A′, B′, C′, are functions of r only, and

$$\phi(r, p) = \mathrm{E}_5 \cos\left(2\sqrt{an(n+1)}\, r^{-s/2}/isp\right) + \mathrm{E}_6 \sin\left(2\sqrt{an(n+1)}\, r^{-s/2}/isp\right)$$

when the negative sign is taken in the above ambiguity, the circular functions being replaced by hyperbolic functions when the positive sign is taken. The value of ip is wholly real when squares of ip may be neglected (*cf.* § 13).

If we deduce the values of B and C from the solution (76), and substitute in the six boundary equations the values so obtained, we shall be left with six linear and homogeneous equations between the six E's. Eliminating the six E's, we have a single relation between n, the constants of the nebula and p. Now it will be seen that it will always be possible to pass to the limit $p = 0$ in this equation, since this amounts only to finding the *ratio* of the values of f_5 or f_6 at the two boundaries. The equation obtained in this manner will give us a knowledge of the configurations at which a change from stability to instability can take place.

§ 19. It therefore appears that it is not sufficient to consider vibrations of frequency $p = 0$ as represented by positions of "limiting equilibrium." The method of POINCARÉ* for determining points of transition from stability to instability is not sufficiently powerful for the present problem; indeed it appears that it is liable to break down whenever there are boundary-equations to be satisfied.†

It is of interest to notice that this complication is not (as might at first sight be suspected) a consequence of our having taken thermal conductivity into account. For we can put $C = 0$ and remove the equation of conduction of heat without causing any change in our argument, except that the right-hand member in equation (74) must be replaced by a determinant consisting only of the minor of the bottom right-hand member in the present determinant. The value of Δ is now

$$\Delta = p^2 D_4 + D_2,$$

and the number of boundary-equations is of course reduced from six to four. Thus an exactly similar situation presents itself, although we are now dealing with a strictly conservative system.

The consequences of this result are more wide-reaching than would appear from the present problem, inasmuch as all problems of finding adjacent configurations of equilibrium are affected. For instance, it appears that an equilibrium theory of tides is meaningless except in very special cases (*e.g.*, when the elements of the fluid in which the tide is raised are physically indistinguishable).

If we attempt to calculate by the ordinary methods the tide raised in a mass of compressible fluid by a small tide-generating potential, we reach a number of equations which are (except in special cases) contradictory. To take a simple case, suppose we have a planet of radius R_0 covered by an ocean of radius R_1, the whole being surrounded by an atmosphere which maintains a constant pressure π at the surface of the ocean. Let the law of compressibility be $\varpi = c\rho$, where c varies from layer to layer of the ocean. Let the tide generating potential be $a_0 r^n S_n$. Then the equations of this paper will hold if we write $p = 0$, $C = 0$, ignore the equation of conduction of heat, replace λT everywhere by c, and include in

* "Sur l'Equilibre d'une Masse fluide . . . ."—'Acta. Math.,' 7, p. 259.

† There is not, of course, a flaw in POINCARÉ'S analysis, but he works on the supposition that the potential-function is a holomorphic function of the principal co-ordinates, and this supposition excludes a case like the present one.

V a term $a_0 r^n$. Equation (39) gives (except in the special case of c = constant), B = 0. Equations (50) and (51) remain unaltered, and give a solution of the form

$$A = E_1 f_1(r) + E_2 f_2(r).$$

Now we must have A = 0 when $r = R_0$, and this determines the ratio E_1/E_2. Also equation (49) must be satisfied, and this leads to a second and different value for E_1/E_2.

A second example, of less interest but greater simplicity, will perhaps help to elucidate the matter. Imagine a non-gravitating medium in equilibrium under no forces inside a rigid boundary. Let the law connecting pressure and density for any particle be $\varpi = \kappa\rho$, where κ varies from particle to particle. In equilibrium ϖ has a constant value ϖ_0. Suppose now that we attempt to find an adjacent configuration which is one of equilibrium under a small disturbing potential V. The general equations of equilibrium are three of the form

$$\frac{dV}{dx} = \frac{1}{\rho}\frac{d\varpi}{dx}.$$

If the position of equilibrium only varies slightly from the initial position, $d\varpi/dx$ will be a small quantity of the first order, so that (to the first order of small quantities) ρ may be replaced by its equilibrium value ϖ_0/κ. We now have

$$\frac{d\varpi}{dx} = \frac{\varpi_0}{\kappa}\frac{dV}{dx},$$

and therefore, since ϖ is a single-valued function of position,

$$\int \frac{1}{\kappa}\frac{dV}{ds}\,ds = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{i.}),$$

the integral being taken along any closed path. Since V and κ are absolutely at our disposal, this equation is, in general, self contradictory. What we have proved is that there will only be an "adjacent" configuration of equilibrium under a potential V if V is a single valued function of κ, a condition which will not in general be satisfied by arbitrary values of V and κ.

It is not difficult to see the physical interpretation of this last result. There were initially an infinite number of equilibrium positions, and therefore an infinite number of vibrations of frequency $p = 0$. To arrive at the configuration of equilibrium under the disturbing force we must imagine vibrations of frequency $p = 0$ to take place until equation (i.) is satisfied; the disturbed configuration will then differ only slightly from the configuration of equilibrium. For instance, if the disturbing field of force consists of a small vertical force g, the fluid must be supposed to arrange itself in horizontal layers of equal density, before we attempt to find the disturbed configuration.

The interpretation of the result obtained in the first instance is similar, but more difficult. Consider a linear series of equilibrium configurations, obtained by the variation of some parameter a, such that the spherical configuration of our example is given by $a = 0$. The other configurations are not symmetrical, the asymmetry being maintained, if necessary, by an external field of force. Every degree of freedom in the configuration $a = 0$ must have its counterpart in the configurations in which a is different from zero. In particular, the principal vibrations of § 12, in which (for the configuration $a = 0$) the dilatation, normal displacement, and temperature-increase all vanish, must have counterparts for all values of a. But when a is different from zero, the above three quantities cannot be supposed to all vanish. In general, therefore, these degrees of freedom provide solutions of the volume-equations, and these solutions contribute to the boundary-equations. In the special case of $a = 0$, these solutions do not affect the boundary-equations at all, so that to rectify the boundary-equations we must, so to speak, take an infinite amount of these solutions. In other words, the complete vibration of frequency $p = 0$ becomes identical with one of the vibrations of § 12, in which u, Δ, and T_1 all vanish.

An Isothermal Nebula.

§ 20. Let us now examine the form assumed by our equations in the simple case in which λ and T are the same at all points of the nebula. We find that, considering only the equations for the case of $p = 0$, equation (39) reduces to

$$C = 0 \quad \ldots \ldots \quad (77),$$

and, in virtue of this simplification, the equation of conduction of heat (36), and the two thermal boundary conditions (33 and 34) are satisfied identically. We are left with equation (55) to be satisfied throughout the gas, and equations (32), (35), (52), and (53) to be satisfied at the boundaries.

The solution of equation (55) is given in equation (56). Now we must satisfy equation (32) by taking $B = 0$ at $r = R_1$, and this, by equation (50), gives the value of A at $r = R_1$ in terms of E_1 and E_2. Hence equation (52) reduces to a homogeneous linear equation between E_1 and E_2.

When n is different from unity, we satisfy equation (35) by taking $A = 0$ at $r = R_0$, and this reduces equation (53) to a homogeneous linear equation between E_1 and E_2.

When $n = 1$, equation (35) reduces to a linear equation between $(A)_{r=R_0}$, E_1 and E_2. Equation (53) is a second equation of the same form, and the elimination of $(A)_{r=R_0}$ from these two equations leads to a homogeneous linear equation between E_1 and E_2.

Thus, in either case, we see that the whole system of equations reduces to a pair of homogeneous linear equations between E_1 and E_2. The elimination of these quantities leaves us with a single equation between n and the constants of the nebula.

We can, therefore, satisfy all the equations for a vibration of frequency $p = 0$ by imposing a single relation upon the constants of the nebula. The unknown solutions which are multiplied by E_5 and E_6 have not been taken into account at all, but since the condition that there shall be a vibration of frequency $p = 0$ must of necessity reduce to a single equation, it will be clear that if these solutions had been taken into account, we should have found it necessary to take $E_5 = E_6 = 0$.

Thus, in the case which we are now considering, a vibration of frequency $p = 0$ is equivalent to a configuration of limiting equilibrium. It is not hard to see that this results from the fact that the particles of which the nebula is composed are physically indistinguishable. This very fact, however, introduces a further complication into the question. It will be noticed that, although the value of ξ has been found at every point of the nebula, it is impossible to determine the separate values of A and B. On the other hand, the physical vibration must have a definite limiting form when $p = 0$. Now it is easy to see that a motion of the gas in which ξ vanishes at every point of the gas, and in which A and B vanish separately at the boundary, will, in every configuration of the gas, satisfy our equations with $p = 0$. Such a motion, in fact, simply leads to a configuration which is physically indistinguishable from the initial configuration, and in which the potential energy remains unaltered. The motion which we have found from our equations is the sum of a motion of this kind, and a true limiting vibration. It is impossible to separate the two motions, except by considering vibrations of frequency different from zero, but fortunately the question is not one of any importance.

§ 21. Let us now attempt to form the final equation in some cases of interest. The equations of an isothermal nebula at rest under its own gravitation have been discussed by Professor DARWIN.* Our function u (equation 54) is given, in the case in which the nebula is isothermal, by the equation

$$u = \frac{2\pi\rho r^2}{\lambda T} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (78),$$

and it will be seen that this is the same as the u of Professor DARWIN's paper. It appears that in general u cannot be expressed as a function of r in finite terms, but a table of numerical values of u is given.† The value of u approximates asymptotically to unity at infinity, so that at infinity ρ varies as r^{-2}. DARWIN's nebula extends from $r = 0$ to $r = \infty$, but it is obvious that we may, without disturbing the equilibrium, replace that part of the nebula which extends from $r = 0$ to $r = R_0$ by a solid core of mass equal to that of the gas which it replaces. We may also remove that part of the nebula which extends from $r = R_1$ to $r = \infty$, if we suppose a pressure to act upon the surface $r = R_1$ of amount equal to the pressure of the gas at this surface. We may suppose the medium outside this surface to be of any kind we please, but as it has already been pointed out that the pressure can, in nature, only be maintained by the impact of matter, we shall suppose that this matter is of a density σ which is continuous with the density ρ of the nebula at the surface of separation. We may now write equation (52) in the simple form

$$\left[\frac{d}{dr}(\xi r^n)\right]_{r=R_1} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (79).$$

We have, up to the present, supposed the nebula to be acted upon by a spherically symmetrical system of forces in addition to its own gravitation. Now it is essential to the plan of our investigation that we shall be able to make the configuration of the nebula vary in some continuous manner, and this compels us to retain this generalisation. We shall, however, suppose that when the nebula extends to infinity, u retains some definite limiting value u_∞, thus including the free nebula as a special case.

* G. H. DARWIN, 'Phil. Trans.,' A, vol. 180, p. 1.

† *Loc. cit.*, p. 15.

§ 22. Let us, in the first place, consider the "series" of nebulæ such that u has a different constant value for each. This series includes a single free nebula, for it appears from DARWIN'S paper that there is a nebula such that $u = 1$ at every point. This nebula, it is true, has infinite density at the centre, but this objection disappears when the innermost shells of gas are replaced by a solid core, the mean density of the core being equal to three times the density of the gas at its surface, and therefore finite. Let us, in the first instance, simplify the problem by supposing that the core is held at rest in space. The boundary equations (35 and 53) which have to be satisfied at $r = R_0$ now take the forms

$$(A)_{r=R_0} = 0 \quad \ldots \ldots \ldots \quad (80),$$

$$\left[\frac{d}{dr}(\xi r^{-(n+1)})\right]_{r=R_0} = 0 \quad \ldots \ldots \ldots \quad (81),$$

independently of the value of n. The value of u in equation (55) being now independent of r, we may write the solution (56) in the form

$$\xi = E_1 r^{\mu} + E_2 r^{\mu'} \quad \ldots \ldots \ldots \quad (82),$$

in which μ, μ' are the roots of the quadratic,

$$t(t-1) = n(n+1) - 2u \quad \ldots \ldots \ldots \quad (83).$$

We accordingly have

$$\mu + \mu' = 1\,;\ \mu - \mu' = 2\sqrt{(n+\tfrac{1}{2})^2 - 2u}\,;\ \mu\mu' = -n(n+1) + 2u \quad . \quad (84).$$

Equation (79) now takes the form

$$E_1(\mu + n)\,R_1^{\mu+n-1} + E_2(\mu' + n)\,R_1^{\mu'+n-1} = 0 \quad \ldots \quad (85),$$

while equation (81) becomes

$$E_1(\mu - n - 1)\,R_0^{\mu-n-2} + E_2(\mu' - n - 1)\,R_0^{\mu'-n-2} = 0 \quad \ldots \quad (86).$$

The elimination of E_1 and E_2 from these equations gives

$$\left(\frac{R_1}{R_0}\right)^{\mu-\mu'} = \frac{(\mu' + n)(\mu - n - 1)}{(\mu + n)(\mu' - n - 1)} \quad \ldots \ldots \ldots \quad (87).$$

The fraction on the right hand can be simplified by the help of equations (84); it is equal to

$$\frac{2(u - (n+\frac{1}{2})^2) + (\mu - \mu')(n + \frac{1}{2})}{2(u - (n+\frac{1}{2})^2) - (\mu - \mu')(n + \frac{1}{2})}.$$

Now the left-hand member of (87) may be replaced by

$$\frac{\cosh\{\frac{1}{2}(\mu - \mu')\log(R_1/R_0)\} + \sinh\{\frac{1}{2}(\mu - \mu')\log(R_1/R_0)\}}{\cosh\{\frac{1}{2}(\mu - \mu')\log(R_1/R_0)\} - \sinh\{\frac{1}{2}(\mu - \mu')\log(R_1/R_0)\}}$$

so that the equation itself reduces to

$$\tanh \{\tfrac{1}{2}(\mu - \mu') \log (R_1/R_0)\} = \frac{\tfrac{1}{2}(\mu - \mu')(n + \tfrac{1}{2})}{u - (n + \tfrac{1}{2})^2} \quad . \quad . \quad . \quad . \quad (88).$$

This equation expresses the relation which must exist between R_1/R_0 and $\mu - \mu'$ (or, what is the same thing, between R_1/R_0 and u), in order that $p = 0$ may be a solution of the frequency equation.

§ 23. We shall be able to interpret this equation most easily by adopting a graphical treatment. If we write

$$x = \tfrac{1}{4}(\mu - \mu')^2, \quad y_1 = -\frac{2(n + \tfrac{1}{2})}{x + (n + \tfrac{1}{2})^2}, \quad y_2 = \frac{1}{\sqrt{x}} \tanh \{\sqrt{x} \log (R_1/R_0)\},$$

then the equation can be written in the form

$$y_1 = y_2.$$

It will be noticed that y_2 remains real when x is negative, an equivalent expression for y_2 being

$$y_2 = \frac{1}{\sqrt{-x}} \tan \{\sqrt{-x} \log (R_1/R_0)\}.$$

The roots of equation (87) are now represented by the intersections of the graphs which are obtained by plotting out y_1 and y_2 as functions of x. These two graphs

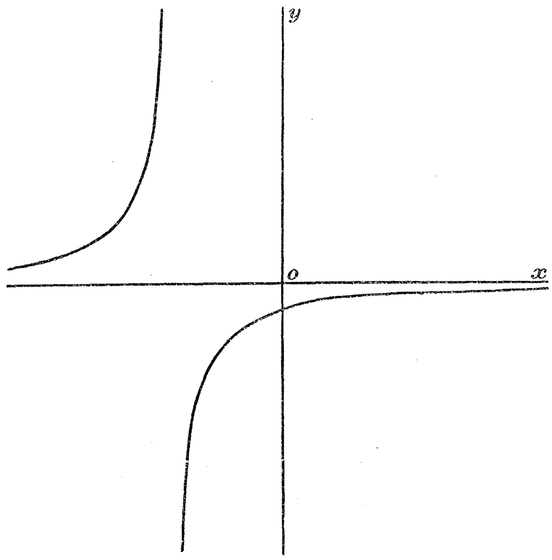

Fig. 1.

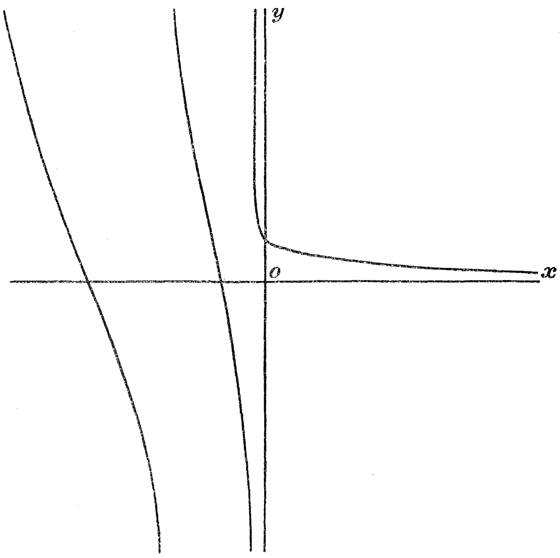

Fig. 2.

are given in figs. 1 and 2 respectively, the graphs being drawn separately for the sake of clearness. The graph for y_1 is, of course, the same for all values of R_1/R_0; that for y_2 can be varied so as to suit any value of R_1/R_0 by supposing it subjected to an appropriate uniform extension parallel to the axis of y, and contraction parallel to the axis of x_1 or *vice versa*. Similarly, different values of $(n + \frac{1}{2})$ can be represented by contraction and extension of the first graph.

If we imagine these two graphs superposed, we see that there cannot, under any circumstances, be an intersection in the region in which x is positive, *i.e.* (equation (84)), for a value of u less than $\frac{1}{2}(n + \frac{1}{2})^2$. The lowest value of u for which an intersection can possibly occur is $u = 1$, and this occurs only when $R_1/R_0 = \infty$. As R_1/R_0 decreases from infinity downwards, the lowest value of u for which an intersection occurs will continually increase. Whatever the value of R_1/R_0 may be, there are always an infinite number of intersections in the region in which $u > \frac{1}{2}(n + \frac{1}{2})^2$.

The values of u found in this way determine the "points of bifurcation" on the linear series obtained by causing u to vary continuously. Thus we have seen that as u continually increases the first point of bifurcation of order n is reached when u has a value which is always greater than $\frac{1}{2}(n + \frac{1}{2})^2$. When R_1/R_0 is very large, the first point of bifurcation is of order $n = 1$, and its position is given by

$$u = 1\tfrac{1}{8} \quad \ldots \ldots \ldots \quad (89).$$

§ 24. Let us, in future, confine our attention to the case in which R_1/R_0 is very large. If we gradually remove the restriction that u is to be independent of r, the various vibrations of frequency $p = 0$ will vary in a continuous manner. Equation (55) remains unaltered in form, and, at infinity, it assumes the definite limiting form

$$r^2 \frac{d^2\xi}{dr^2} = \{n(n+1) - 2u_\infty\}\xi \quad \ldots \ldots \quad (90),$$

where u_∞ is the limit (supposed definite) of u at infinity. It therefore appears that at infinity the solution for ξ approximates asymptotically to that given by equation (82), if μ, μ' are now taken to be the roots of

$$t(t-1) = n(n+1) - 2u_\infty \quad \ldots \ldots \quad (91).$$

Equation (85) accordingly remains unaltered. Equation (81) takes a form which is no longer represented by equation (86), but which will impose some definite ratio upon E_1 and E_2. It is therefore clear that when R_1 is very great, equation (85) can only be satisfied, at any rate so long as μ and μ' are real, by taking $\mu - \mu'$ very small. Thus a point of bifurcation will again be given by $\mu - \mu' = 0$, our previous investigation sufficing to show that this gives a genuine solution to our problem, and does not correspond to an irrelevant factor introduced in the transformation of our equations. This point of bifurcation is moreover the first one reached as u increases, since it is at the point at which μ, μ' change from being real to being complex.

We conclude that, independently of the values of u at points inside the nebula, the smallest value of u_∞ for which a vibration of zero frequency and of order n is possible is given by

$$u_\infty = \tfrac{1}{2}(n + \tfrac{1}{2})^2 \quad \text{(92)},$$

or, for all orders, is given by

$$u_\infty = 1\tfrac{1}{8} \quad \text{(93)},$$

the limiting vibration being of order $n = 1$.

It ought to be noticed that for this limiting vibration equation (82) fails to represent the solution owing to μ and μ' becoming identical. The true solutions for real, zero, and imaginary values of $\mu - \mu'$ may be put respectively in the forms

$$\xi = C_1\sqrt{r}\sinh\{\tfrac{1}{2}(\mu - \mu')\log \epsilon R\},$$

$$\xi = C_1\sqrt{r}\log \epsilon R,$$

$$\xi = C_1\sqrt{r}\sin\left\{\frac{i}{2}(\mu - \mu')\log \epsilon R\right\},$$

in which C_1 and ϵ are constants of integration.

At infinity ρ vanishes to the order of $1/r^2$, so that $d\rho/dr = -2\rho/r$. The value of ξ for very great values of r is therefore (equation (50))

$$\xi = \lambda T(-2A + Br).$$

At the outer boundary a surface-equation (32) directs us to take $B = 0$. Following this out, we find that at infinity A is of the same order as ξ, and therefore becomes infinite to the order of $\sqrt{r}$. Suppose, on the other hand, that we start by taking $A = 0$, so that $B = \xi/\lambda T r$. The value of B now vanishes at infinity to the order of $1/\sqrt{r}$, and the surface-equation (32) is satisfied by a motion which vanishes at infinity. It would therefore appear to be easier to satisfy the boundary conditions when r is actually infinite than when r is merely very great. This result opens up a somewhat difficult question, which will be considered in the next section.

Before passing on, we may consider in what way the results which have already been obtained will be modified, if we suppose the core of the nebula to be free to move in space, instead of being held fast. For the free nebula $u_\infty = 1$, so that our results show that a free nebula will be stable if the core is supposed fixed. The same must therefore obviously be true when the core is free to move, since a motion in which nebula and core move as a single rigid body will not influence the potential energy. When the nebula is not free, fixing the core may be regarded as imposing a constraint which does no work; freedom of the core therefore tends towards instability. It will be proved in § 28, that a nebula is stable for values of u_∞ which are less than the critical value, and unstable for values greater than this value. Assuming this for the moment, we see that a nebula in which the core is free to move must necessarily be unstable if u_∞ has a value greater than $1\frac{1}{8}$.

If then, we start with a free nebula and imagine u_∞ to gradually increase from $u_\infty = 1$ upwards, the core being free, it follows that the nebula will first become unstable when u_∞ reaches some value such that

$$1\tfrac{1}{8} > u_\infty > 1 \quad \ldots \quad (94).$$

§ 25. The nebula extending to infinity, let us attempt to find the displacement which will be caused by a small disturbing potential v_n given by

$$v_n = \frac{4\pi}{2n+1}\left\{\frac{a_0}{r^{n+1}} + a_1 r^n\right\} S_n \quad \ldots \quad (95).$$

It is clear that the displacement required will be given by our equations if we include in V (equation (27)) the terms

$$\frac{4\pi}{2n+1}\left\{\frac{a_0}{r^{n+1}} + a_1 r^n\right\}.$$

The equation replacing (42) may be transformed in the manner of § 15, and the resulting equations will be those of § 16, except that we must replace (52) by

$$\left[\frac{1}{r^{2n}}\frac{d}{dr}(\xi r^n)\right]_{r=R_1} = a_1 - \left[\frac{A(\rho - \sigma_1)}{r^{n-1}}\right]_{r=R_1} \quad \ldots \quad (96),$$

and (53) by a similar equation.

If a displacement can be found to satisfy these modified equations, the external disturbing potential which will be required to hold the system in this displaced position will be given by equation (94). Now the condition that this displaced position shall be one of limiting equilibrium is that this disturbing potential must vanish. To be more precise, v_n must be such that the force derived from it vanishes at every point of the nebula. We must therefore have

$$a_1 r^n = 0$$

at all points of the nebula, including $r = R_1$. Now (95) may be regarded as an equation giving a_1 in terms of R_1. Taking $\rho = \sigma_1$, as before, we find from (95)

$$a_1 r^n = \left(\frac{r}{R_1}\right)^n \left[\frac{1}{R_1^n}\frac{d}{dr}(\xi r^n)\right]_{r=R_1},$$

and this vanishes at all points, including $r = R_1$, in the case in which R_1 is put equal to infinity, if

$$\underset{r=\infty}{\mathrm{Lt}} \frac{\xi}{r} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (97).$$

The condition that a_0/r^{n+1} shall vanish at every point would lead to a similar equation to be satisfied at the origin, if there were no core. If, however, we retain the core, it leads to the same equation as was found in § 22 (equation (81), when the core is held at rest). Thus, our present method of finding a position of limiting equilibrium has led to a result different from that obtained by the search for a vibration of zero frequency, in that equation (97) replaces equation (79).

The value of ξ at infinity is given by equation (82); hence we have

$$\underset{r=\infty}{\mathrm{Lt}} \frac{\xi}{r} = [E_1 r^{\mu-1} + E_2 r^{\mu'-1}]_{r=\infty} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (98).$$

As before, the equation to be satisfied at $r = R_0$ determines the ratio of E_1 to E_2: equation (97) is therefore satisfied if the real parts of μ and μ' are each less than unity. Now μ, μ' are the roots of equation (91), hence this condition is satisfied provided

$$n(n+1) < 2u_\infty \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (99).$$

§ 26. Let the kinetic and potential energies of a small displacement be given, in terms of the principal co-ordinates, by

$$2T = a_1 \dot{x}_1^2 + a_2 \dot{x}_2^2 + \ldots$$
$$2V = b_1 x_1^2 + b_2 x_2^2 + \ldots$$

so that the equations of motion are

$$a_1 \ddot{x}_1 - b_1 x = 0$$

&c., and p^2 is given by

$$a_1 p^2 = b_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (100).$$

The method of §§ 20–24 amounted to finding vibrations such that $p^2 = 0$, and therefore, by equation (100), solutions of

$$b_1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (101).$$

In § 25, on the other hand, we started with the supposition that the nebula extended to infinity, so that all the quantities a and b are liable to become infinite. The equation giving vibrations of frequency $p = 0$ is no longer equation (101), but is

$$\underset{R_1=\infty}{\mathrm{Lt}} \frac{b_1}{a_1} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (102),$$

and this is obviously more general than equation (101).

It will be noticed that the method of §§ 20–24 is the method which is mathematically appropriate to the case of a nebula enclosed in a surface maintained at constant pressure, while the method of § 25 is that appropriate to an infinite nebula. In the former case, a vibration of frequency $p = 0$ may represent a real change from stability to instability; in the latter case such a vibration leads to an adjacent configuration of equilibrium, and is, in this sense, a point of bifurcation, but does not denote a change in the sign of p^2.

The General Case of a Nebula extending to Infinity.

§ 27. The method to be followed has been explained in § 18. The general differential equation is of the sixth order. Four solutions have definite limiting forms when $p = 0$; the remaining two take singular forms. The former have been examined in § 16; the latter are represented mathematically (p. 18) by functions which do not approach a definite limit as p approaches a zero value, and physically (p. 11) by systems of steady currents.

There are six constants of integration, E_1, E_2, E_3, E_4, E_5, E_6, of which the two last belong to the singular solutions. Let us suppose (as is always possible (p. 19)) that the ratios of these six constants are determined from five of the boundary-equations, that which is not used being the equation satisfied by ξ at the outer boundary. This remaining boundary-equation now takes the form (*cf.* equation (56))

$$E_1\psi_1(R_1) + E_2\psi_2(R_1) + E_5\psi_5(R_1) + E_6\psi_6(R_1) = 0 \quad . \quad . \quad . \quad (103),$$

in which the four E's are definite quantities. The four ψ's must have definite limiting values (zero and infinity being included as possible values) when $R_1 = \infty$. Thus in equation (103) some terms must preponderate over the others. When the nebula is isothermal, these terms are the first two. Hence, when the nebula is not isothermal, it follows from the principle of continuity, that the same two terms must still preponderate, at any rate for some finite domain including the isothermal nebula. Otherwise it would be possible to change the stability or instability of a nebula by an infinitesimal change in the physical constitution of the nebula. Hence throughout this domain, equation (103) must reduce to its first two terms, *i.e.*, must become formally the same as in the case of the isothermal nebula. But the solution for ξ (and therefore the functions ψ_1, ψ_2), remain formally the same in the general case as in this particular case, and therefore the stability-criterion derived from equation (103) remains formally the same.

It follows that whether the nebula is isothermal or not (provided always that the configuration lies within a certain domain of equilibrium configurations) the critical configurations are given by the two equations (92) and (99).

Exchange of Stabilities.

§ 28. We have now completed an investigation of the configurations at which a transition from stability to instability can occur, as regards the spherical form, for vibrations of orders different from zero. It is unnecessary to discuss vibrations of order $n = 0$, for the following reason.

Our problem is to determine the changes in the configuration of a nebula which will take place as the nebula cools, starting from a spherical configuration, supposed stable. We are not concerned with the succession of spherical configurations, but only with an investigation of the conditions under which a spherical configuration becomes a physical impossibility. Now a point of bifurcation of order $n = 0$ does not indicate a departure from the spherical configuration. It indicates a choice of two paths, one stable and the other unstable, and the configurations on both paths will remain spherically symmetrical.

We have therefore determined already the circumstances under which a transition from a symmetrical to an unsymmetrical configuration can occur. It remains to show that there is, in effect, an exchange of stabilities at a point of bifurcation, and to examine on which side of the point of bifurcation the spherical configuration is stable.

We are going to prove that the spherical configuration is stable for all values of u less than u_0, the lowest value of u at which a point of bifurcation of order different from zero can occur. Our method will be as follows: Any two equilibrium configurations can be connected by a continuous linear series of equilibrium configurations, and u will vary continuously as we move along this series. If one of the two terminal configurations is stable, and if the linear series can be chosen so that u does not at any point of it pass through a value for which a vibration of frequency $p = 0$ is possible, then we know that the other terminal configuration is also stable.

The value of γ, the gravitation constant, has been taken equal to unity. If this constant is restored, the value of u becomes (equation (54))

$$u = 2\pi\gamma\rho r^2 \frac{d\rho}{dr} \Big/ \frac{d\varpi}{dr}.$$

Since $\varpi = \lambda T\rho$, we have

$$\frac{d\varpi}{dr} = \rho \frac{d}{dr}(\lambda T) + \lambda T \frac{d\rho}{dr}.$$

For an infinite nebula, the first term on the right-hand of this equation will vanish at infinity in comparison with the second. Hence we have as the value of u_∞

$$u_\infty = \operatorname*{Lt}_{r=\infty} \frac{2\pi\gamma\rho r^2}{\lambda T} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (104).$$

If we write $\gamma = 0$ we pass to the case of a non-gravitating nebula, and we see that $u_\infty = 0$ provided the ratio of ρr^2 to λT remains finite at infinity. Now we can keep the value of ρ and λT the same at every point by subjecting the nebula to an appropriate external field of force, and this field of force will be exactly the same as the gravitational field which was annihilated upon putting $\gamma = 0$. It is spherically symmetrical, and its potential vanishes at infinity to the order of $1/r$, so that it comes within the scope of our previous analysis. For values of γ intermediate between the natural value ($\gamma = 1$) and the value $\gamma = 0$ we can obtain the same result by taking a field of force equal to $1 - \gamma$ times the foregoing. As we increase γ from 0 to 1 we obtain a linear series, in which the configuration of the nebula is unaltered, the nebula being gradually endowed with the power of gravitation.

For the general configuration of this series, consider the work done in a specified displacement, which is proportional to S_n at every point. The potential (gravitational + that of external field) after displacement will be of the form

$$a + b\gamma S_n,$$

where a and b are functions of r and independent of γ. The total work done against this field during the displacement is therefore of the form

$$B\gamma,$$

where B is independent of γ and depends solely upon the particular displacement selected. The work done against the elastic forces is of course independent of γ, and depends solely upon the displacement selected. This work is essentially positive. The total work is therefore of the form

$$A + B\gamma,$$

where A is positive and B may (§ 2) be negative. Since γ is proportional to u_∞ this may be written

$$A + B'u_\infty \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (105).$$

Suppose this function calculated for all possible displacements. Then we shall find that for values greater than some definite value of u_∞ it is possible for the work done to become negative. For values of u_∞ less than this critical value, the work will be positive for all displacements. Hence from the form of expression (105) it follows that the passage of u_∞ through a critical value denotes a real change from stability to instability, and that the stable configurations are given by the smaller values of u_∞.

Recapitulation and Discussion of Results.

§ 29. We have seen that the vibrations of any spherical nebula may be classified into vibrations of orders $n = 0, 1, 2$, &c., a vibration of any order n being such that the displacement and change in temperature at any point are each proportional to some spherical surface harmonic S_n of order n. The frequency of vibration is independent of the particular spherical harmonic chosen, depending only upon the order n.

The vibrations of order $n = 0$ have been seen to be of no importance; the stability of the vibrations of orders different from zero has been discussed, in the limiting case in which the nebula extends to infinity, with the following results:—

Starting from any stable configuration of spherical symmetry, the vibrations of any order n, different from zero, all remain stable until the function u_∞, defined by equation (104), passes through a certain critical value. In any case this critical value is first attained for a vibration of order $n = 1$.

For a nebula which actually extends to infinity, the critical value is $u_\infty = 1$. When this value is reached we come to a second series of equilibrium configurations, the form of which will be investigated later. If this value is passed, the configuration remaining spherical, there will not be vibrations in which the time enters through a real exponential factor, because the critical vibrations remain of frequency $p = 0$, the inertia of the nebula being infinite.

If the radius R_1 of the nebula is regarded as very great but not infinite, this statement is not true, since the inertia cannot now become infinite. In this case the first new series of equilibrium configurations is again reached when $(u)_{r=R_1}$ attains a certain critical value, and the critical vibration is again of order $n = 1$. The critical value of $(u)_{r=R_1}$ has not been calculated, but when R_1 becomes infinite, it has a limiting value which has been shown to lie between 1 and $1\frac{1}{8}$.

Taking $\gamma = 1$, we have as the value of u_∞,

$$u_\infty = \underset{r=\infty}{\mathrm{L}^t} \frac{2\pi\rho r^2}{\lambda T} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (106).$$

The question of stability turns entirely upon the value of this function, which may appropriately be termed the "stability-function."

We now see that the whole question of stability depends upon the ratio of the density to the elasticity at infinity. This result is not hard to understand. In the first place, since the nebula extends to infinity, we may, so to speak, measure it upon any linear scale we like. If we measure it on a sufficiently great scale, the nebula still remains of infinite extent, but the variations in temperature or structure which occur near the centre can be made to appear as small as we wish, and the solid core can be made to appear as insignificant as we wish. Thus by measuring any nebula upon a sufficiently great scale we can make it appear indistinguishable from an isothermal nebula, and the critical vibration for which $p = 0$ does not disappear from sight, since in the limit this vibration (measured by ξ/r) remains finite at infinity. Further, as Professor DARWIN points out, we can make it appear like a nebula in which u maintains a constant value throughout.* Passing on, we notice that the stability function now depends solely upon the ratio of the density to the elasticity. The different elements of the nebula are attracted towards one another by their mutual gravitation, and are kept apart by the elasticity of the gas. For certain values of the ratio of these two systems of forces, it will be possible to find displacements in which the work done by one system exactly balances that done against the other, and these are the critical vibrations.

The stability function u_∞ is a function only of the quantities determining the equilibrium configuration of the nebula, and its value may therefore be found from the equations of equilibrium. We proceed to examine the value.

Evaluation of the Stability Function.

General Case of a Nebula at Rest.

§ 30. We have already quoted Professor Darwin's result that $u_\infty = 1$ for an isothermal nebula at rest, and the considerations put forward in the last section will probably suggest that the result in the more general case will be found to be independent of variations in temperature at finite distances, provided only that the temperature has a definite limit at infinity. We shall, however, examine the question *ab initio*, using a slight modification of Darwin's method, and making the problem more general by retaining a spherically symmetrical system of external forces.

We shall denote the potential of this system of forces by V′, and use V to denote the gravitational potential of the nebula itself. The total potential is now V + V′, so that the equation of equilibrium, equation (11), takes the form

$$\frac{1}{\rho}\frac{d}{dr}(\lambda T\rho) - \frac{d}{dr}(V + V') = 0,$$

and if M is the mass of the solid core, this can be written

$$\frac{r^2}{\rho}\frac{d}{dr}(\lambda T\rho) + 4\pi\int_{R_0}^{r}\rho r^2\,dr + M - r^2\frac{dV'}{dr} = 0 \quad . \quad . \quad . \quad . \quad (107).$$

Differentiating with respect to r,

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{d}{dr}(\lambda T\rho)\right) + 4\pi\rho r^2 - \frac{d}{dr}\left(r^2\frac{dV'}{dr}\right) = 0.$$

Write

$$\lambda T\rho = e^y,$$

* G. H. Darwin (*l.c. ante*, p. 16), "If we view the nebula from a very great distance, . . . the solution of the problem becomes $y = \log 2x^2$." Now $u = -\frac{1}{2}x^2\,d^2y/dx^2$, so that this solution is equivalent to $u = 1$. This justifies our statement, and shows at the same time that for any nebula at rest and in equilibrium u_∞ has the critical value $u_\infty = 1$, provided it is acted upon by no forces except its own gravitation.

and

$$x = \int_r^\infty \frac{dr}{\lambda T r^2},$$

so that

$$\frac{d}{dr} = \frac{1}{\lambda T r^2}\frac{d}{dx},$$

then the above equation may be written

$$\frac{d^2y}{dx^2} + 4\pi e^y r^4 - r^2 \frac{d}{dr}\left(r^2 \frac{dV'}{dr}\right) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (108).$$

At infinity we are supposing λT to have a definite and finite limit, so that the limiting value of x is $1/\lambda T r$. Let us further suppose that $\frac{d}{dr}\left(r^2\frac{dV'}{dr}\right)$ has a definite limit given by

$$\frac{d}{dr}\left(r^2 \frac{dV'}{dr}\right) = V'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (109),$$

and that squares of V'' may be neglected. Then the limiting form of (108) at infinity is

$$\frac{d^2y}{dx^2} + \frac{4\pi e^y}{(\lambda T x)^4} - \frac{V''}{(\lambda T x)^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (110).$$

Write

$$y = \eta + \log \frac{\lambda^4 T^4 x^2}{2\pi},$$

then

$$\frac{d^2y}{dx^2} = \frac{d^2\eta}{dx^2} - \frac{2}{x^2} + 4\frac{d^2}{dx^2}\log(\lambda T),$$

and

$$\frac{4\pi e^y}{(\lambda T x)^4} = \frac{2e^\eta}{x^2}.$$

Equation (110) is now transformed into

$$\frac{d^2\eta}{dx^2} + \frac{2}{x^2}(e^\eta - 1) + 4\frac{d^2}{dx^2}\log(\lambda T) - \frac{V''}{(\lambda T x)^2} = 0 \quad . \quad . \quad . \quad . \quad (111).$$

In the special case in which $\frac{d^2}{dx^2}\log(\lambda T)$ vanishes, this may be written

$$\frac{d^2\eta}{dx^2} + \frac{2}{x^2}\left(\eta + \tfrac{1}{2}\eta^2 + \tfrac{1}{6}\eta^3 + \ldots - \frac{V''}{2\lambda^2 T^2}\right) = 0,$$

and at infinity (*i.e.*, for very small values of x), the solution is

$$\eta = \frac{V''}{2\lambda^2 T^2} + \sqrt{\frac{x}{A}}\cos\left(\tfrac{1}{2}\sqrt{7}\log\frac{x}{B}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (112)$$

where A, B are the two constants of integration.

In the more general case in which $\frac{d^2}{dx^2}\log(\lambda T)$ cannot be supposed to vanish, it is clear that this term will vanish at infinity in comparison with the other terms in (111), if η has the limiting value given by (112), and therefore that (112) is the limit, at infinity, of the solution of (108).

Of the two arbitrary constants, A and B, the former corresponds to the indeterminateness of the linear scale upon which the nebula is measured, the second to the indeterminateness of the conditions at the inner surface of the nebula. If there is no core, there is only one value of A/B which will give a finite density of matter at the centre of the nebula. Further information as to equilibrium configurations can be found in Professor DARWIN's paper, or in a paper by A. RITTER.*

For our purpose it is sufficient to know that the second term in η vanishes with x for all values of A and B. Hence at infinity

$$y = \frac{V''}{2\lambda^2 T^2} + \log\frac{\lambda^4 T^4 x^2}{2\pi},$$

$$\rho = \frac{e^y}{\lambda T} = \frac{\lambda^3 T^3 x^2}{2\pi} e^{V''/2\lambda^2 T^2} = \frac{\lambda T}{2\pi r^2} e^{V''/2\lambda^2 T^2},$$

and hence (equation (107))

$$u_\infty = \frac{2\pi\rho r^2}{\lambda T} e^{V''/2\lambda^2 T^2} = 1 + \frac{V''}{2\lambda^2 T^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (113).$$

Putting $V'' = 0$, we arrive at the anticipated result that the stability function has a unit value, for every nebula which extends to infinity in such a way that λT has a finite limit at infinity.

A Slowly Rotating Nebula.

§ 31. The case which is of the greatest physical interest, is that in which the nebula is not at rest but is rotating in a position of relative equilibrium

Here the arrangement is no longer in spherical shells, so that the foregoing analysis breaks down. If, however, we suppose the rotation ω to be so small that ω^4 may be neglected, it will be easy to modify the foregoing analysis, so as to take account of rotation.

We shall still suppose the nebula to extend to infinity, so that we must not suppose the rotation to be the same at all distances, for in this case a finite value of ω would imply an infinite velocity of those parts of the nebula which are at infinity. Let us

* 'Wied. Ann.,' vol. 16, p. 166.

suppose that at infinity the linear velocity approximates to a finite limit, so that we may write

$$\omega = \Omega/r$$

for all values of r greater than a certain amount.*

So long as we are only concerned with configurations of equilibrium and vibrations of frequency $p = 0$, the rotation may be allowed for by the introduction of a force of amount $\omega^2 r \sin\theta$ per unit mass, acting perpendicular to the axis of rotation; or, what comes to the same thing, by the introduction of a potential

$$\tfrac{2}{3}(1 - P_2)\int^r \omega^2 r\, dr,$$

or

$$(1 - P_2)\, V'$$

where, for all values of r greater than a certain value,

$$V' = \tfrac{2}{3}\Omega^2 \log r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (114).$$

Let us examine separately the two effects arising from the two terms of this potential, beginning with the term $-P_2 V'$. There will in this case be a correction to be applied to all equations, and this correction will consist of the addition of a small term containing $\omega^2 P_2$. Let us suppose that all symbols which have so far denoted functions of r, denote in future the mean value of the corresponding quantities averaged over a sphere of radius r. For instance, ρ is no longer the density at distance r from the centre, but is the mean density over the sphere of radius r. The density at any point will be of the form $\rho + \omega^2 P_2 \rho_2$, where ρ_2 is a function of r. We may in every case equate the coefficients of different harmonics, and by equating the coefficients of terms which do not contain the terms $\omega^2 P_2$, we shall obtain the same equations as were obtained in the case of $\omega = 0$, except that the meaning of every term is altered.

The equations derived from the parts which do not contain ω will suffice, as before, to determine p, so that the values of p are of the same form as before, except that the quantities involved have a slightly different meaning. Hence the stability criterion is still given by the value of the stability function u_∞; while equation (107)

* This particular law is chosen for examination because it leads most quickly to the required result. The case in which ω vanishes at infinity more rapidly than $1/r$ is covered by taking $\Omega = 0$. Heie, however, the angular momentum vanishes in comparison with the mass, and it is not surprising to find that a rotation of this kind does not affect the question of stabitity. The case in which ω vanishes less rapidly than $1/r$ is physically impossible, since it gives an infinite linear velocity at infinity, but may be theoretically included in the case of $\Omega = \infty$.

Any special assumption about the value of ω at infinity must, however, disappear when we turn to the case of a finite nebula (§ 26), in which Ω may be appropriately supposed to correspond to the surface velocity ωR_1.

remains true, if the new meaning is given to the symbols in each case. We conclude that the question of stability is not affected by the potential $-P_2V'$.

The remaining potential term is the spherically symmetrical term V'. The total potential may now be taken to be $V+V'$, and this potential, besides being spherically symmetrical, satisfies the condition which was postulated in the determination of the criterion of stability; namely, that its radial differential coefficient shall vanish at infinity to the order of $1/r$. The value of the derived function V'' (equation (109)) is

$$V'' = \underset{r=\infty}{\mathrm{Lt}} \frac{d}{dr}\left(r^2 \frac{dV'}{dr}\right) = \tfrac{2}{3}\Omega^2, \text{ by equation (114).}$$

Hence the stability function is given by (*cf.* equation (113))

$$u_\infty = 1 + \frac{\Omega^2}{3\lambda^2 T^2}.$$

We have therefore found that when an infinite nebula is rotating, with such angular velocities that the linear velocities at infinity have the limiting value Ω, the value of u_∞ is greater than unity no matter how small Ω may be. This result has only been obtained on the supposition that ω^4 may be neglected. We have obtained no information as to what happens when ω^4 is taken into account, *i.e.*, when the square of the "ellipticity" of the nebula is taken into account.

Influence of Viscosity.

§ 32. No account has so far been taken of the viscosity of the gas. The terms arising from viscosity which may be supposed to occur in the true equations of motion, will contain the coefficient of viscosity (μ), and will in each case depend on velocities and not on displacements. Hence viscosity enters the equations of motion through the factor μip. The vibrations for which $p = 0$ are accordingly unaffected by viscosity, and since it is upon the existence of such vibrations that the whole question of stability turns, it is clear that the results already obtained must remain true even in the presence of viscosity.

It can be shown that equations (24) to (26) specify a principal vibration, whether the gas is viscous or not. The result is stated without proof, as the proof is rather lengthy, and has no bearing upon the main question under discussion.

A Nebula in Process of Cooling.

§ 33. In the mathematical investigation we have been concerned with vibrations about a position of absolute equilibrium. In nature, no such position of absolute equilibrium will occur; the condition of the nebula will be incessantly changing.

Let us suppose the temperature of the nebula to be continually cooling, owing either to radiation of heat from its surface or to a process of quasi-evaporation such as is described in Professor DARWIN's paper (§ 13 or p. 66). Since the gas (or quasi-gas) is not a perfect conductor, the nebula will not at any time be in perfect thermal equilibrium. The changes in density of all parts, and in the temperature of the inner parts of the nebula will, so to speak, lag behind their equilibrium values as determined by the changes in the temperature of the outer part of the nebula. It is, therefore, clear that so long as the nebula is cooling, the ratio of the density to elasticity in the outermost layers of gas will be greater than that calculated upon the assumption of perfect equilibrium. This "lag" accordingly decreases the value of the stability-function, and so supplies a factor which tends to instability.

SUMMARY AND DISCUSSION OF RESULTS.

§ 34. Let us now examine to what extent we have found solutions of the two problems propounded in § 4.

Firstly, as regards the stability of a spherical nebula of very great size, of which the outer surface is maintained at constant pressure. We have found that the stability-function for such a nebula (in the limiting case in which the outer radius is infinite) has a unit value when the nebula is in equilibrium and at rest. This value is increased by allowing for the "lag" in temperature caused by the cooling of the nebula. It is also increased by a rotation of the nebula, at any rate so long as this rotation is small. The nebula will become unstable as soon as the stability-function becomes greater than a certain value, which has not been calculated, but is known to be between 1 and $1\frac{1}{8}$. The investigation of § 23 leads us to expect that the critical value of the stability-function will increase as R_1 decreases, although this has only been strictly proved for a single case.

It is therefore possible that, even when the nebula is non-rotating, the temperature-lag may be sufficient to make the nebula unstable. If we disregard the temperature-lag, it seems probable that a small rotation will suffice to bring about instability. This latter question, however, deserves more detailed examination.

§ 35. Let us suppose that the nebula starts from rest in a configuration of absolute equilibrium, and that the rotation is gradually increased. In this way we obtain a linear series of configurations of relative equilibrium. When the rotation is small, the configuration, instead of being strictly spherical is slightly spheroidal. The series we are considering is therefore the analogue of the series of MACLAURIN spheroids of an incompressible fluid. So long as the rotation remains small, we may separate the two terms of the rotation-potential in the manner explained in § 31. We may, in fact, suppose our analysis still to apply as if the configuration remained spherical, and the only effect of the rotation is to increase the value of the stability-function. For larger values of ω^2, all our results are subject to a correction of the order of ω^4. For small values of ω^2, the value of ω^2 will be proportional as we have seen (§ 31) to $u_\infty - 1$, so that this correction may be supposed to be proportional to $(u_\infty - 1)^2$. The first points of bifurcation of orders 1, 2 occur (in the spherical configuration) at $u_\infty - 1 = \theta$, $2\frac{1}{8}$ respectively, where θ is known to be less than $\frac{1}{8}$. Now it would seem to be fairly safe to neglect θ^2, but even if we waive this point, it will be admitted that the correction of the order of $(u_\infty - 1)^2$ cannot be so great as to change the order in which these two points of bifurcation will occur.

We therefore see that a rotating nebula will become unstable for a comparatively small value of ω^2, the critical vibration being of order $n = 1$. The new linear series is one in which (except for the spheroidal deformation caused by the rotation) the surfaces of equal density remain spheres, which are no longer concentric. The linear series of order $n = 2$ will accordingly be unstable: this is the analogue to the series of Jacobian ellipsoids in the incompressible fluid.

§ 36. The case of a nebula which actually extends to infinity is much simpler. Here the value of u_∞ is again unity, and this value is increased, as before, either by temperature-lag or rotation. Every point at which u_∞ is greater than unity is in one sense a point of bifurcation, since starting from this point there is a series of unsymmetrical equilibrium configurations. Strictly speaking, these points do not indicate an exchange of stabilities, for the critical vibrations remain of frequency $p = 0$ even after passing the point. They possess, however, the property that a critical vibration, if once started, will continue increasing, since the forces of restitution (of whichever sign) vanish in comparison with the momentum of the vibration.

§ 37. Let us now try and examine which of these two hypotheses is best capable of representing the "primitive nebula" of astronomy. Imagine a sphere S drawn in the nebula, the radius being a, and the pressure at this surface π. The matter inside S is to form a spherical nebula of finite extent, bounded by a sphere over which the pressure is π, and this matter is to be of a density sufficient to warrant us in assuming the gas-equations at every point. The surface S will be continually traversed by matter, but this will be of no consequence if the losses and gains balance in every respect. The matter outside S must supply the pressure π, and will also, as was explained in the introduction (§ 3), influence the matter inside S by its motion.

Imagine the matter inside S to be executing a small vibration, and consider two extreme hypotheses as to the behaviour of the matter outside S.

Suppose, in the first place, that the matter outside S is such that it and the matter inside S together form a perfect spherical nebula at rest. Then the motion of the matter outside S is given by the equations of vibration of such a nebula, and the influence of this matter upon that inside S is exactly that required in order to enable the matter inside S to execute the vibrations given by the equations of an infinite nebula.

Suppose, next, that the matter outside S consists mainly of molecules or of masses of matter which are describing hyperbolic or parabolic orbits, or which come from infinity and after rebounding from the nebula return to infinity. Suppose, further, that the interval during which such a mass is appreciably under the influence of the matter inside S is so small that it is not appreciably affected by the motion of the latter. In this case the matter outside S may be regarded as arranged at random, independently of the vibrations of the matter inside S; it will not, as under our first supposition, take up the motion of the matter inside S to any appreciable extent. Hence the matter outside S will exert no force upon that inside S except the constant pressure π, and the vibrations of the matter inside S will be those of a spherical nebula of finite size, bounded by a surface at constant pressure π.

These two extreme hypotheses lead, as we can now see, to the two conceptions of a nebula put forward in § 4. In nature the truth will lie somewhere between these two hypotheses, and it is by no means easy to decide which of the two gives the better representation of an actual nebula. We shall, however, be within the limits of safety if we assert of an actual nebula only those propositions which are true of both our ideal nebulæ.

§ 38. We may accordingly sum up as follows :—

(i.) A nebula at rest and in absolute equilibrium in a spherical configuration will always be stable.

(ii.) Such a nebula may become unstable as soon as the temperature-lag is taken into account.

(iii.) There will be a linear series of configurations of relative equilibrium of a rotating nebula, starting from a non-rotating spherical nebula (supposed stable), and such that the configuration is symmetrical about the axis of rotation. This linear series corresponds to the series of Maclaurin spheroids.

(iv.) The first point of bifurcation on this series occurs for a comparatively small value of the angular rotation.

(v.) The second series through this point is one in which the configurations possess only two planes of symmetry. Initially the configuration is such that the equations to the surfaces of equal density contain only terms in the first harmonic in addition to those required by the angular rotation.

(vi.) There is a linear series which corresponds to the series of Jacobian ellipsoids, each configuration possessing three planes of symmetry. The point of bifurcation at which this series meets the series mentioned in (iii.) is a point at which the angular rotation is much larger than that at the point of bifurcation mentioned in (iv.).

(vii.) This latter linear series appears to be always unstable.

THE UNSYMMETRICAL CONFIGURATIONS OF A NEBULA.

The Second Series of Equilibrium Configurations.

§ 39. Let us now try to examine the second series of equilibrium configurations, which, as we have seen, is a series of stable configurations replacing the series of Jacobian ellipsoids. In this way we shall be able to gather some evidence with a view to forming a judgment whether the behaviour of the nebula after leaving the symmetrical configuration is such as is required by the nebular hypothesis.

Let us suppose, in the first instance, that the symmetrical configuration from which this series starts is one in which there is no rotation, so that the configuration is one of perfect spherical symmetry. If the nebula is one in which cooling takes place very slowly, the configuration of the nebula will always be very approximately an equilibrium configuration. This configuration will be one of the spherically symmetrical series until the first point of bifurcation is reached; after this the configuration will change so as to move along the other series, which passes through this point.

Now we have already found the manner in which the configuration first diverges from spherical symmetry; in other words, we have a knowledge of the unsymmetrical series in the immediate neighbourhood of the point of bifurcation. If then, we can, by some method of continued approximation, obtain a more extended knowledge of this series of configurations, we shall be able to trace the motion of a nebula which is cooling with infinite slowness, and in this way form some idea of the motion to be expected in the more general case.

Let us assume, as a general form for the "series" now under discussion,

$$\rho = \rho_0 + \rho_1 P_1 + \rho_2 P_2 + \rho_3 P_3 + \dots \qquad (115),$$

where P_s is the zonal harmonic of order s, and ρ_0, ρ_1, ρ_2 are functions of r and of some parameter a. This parameter determines the position of any particular configuration in the series. We shall suppose that at the point of bifurcation $a = 0$, and we then know that when a is very small the limiting form of ρ is

$$\rho = \rho_0 + \rho_1 P_1.$$

In the notation which has been in use throughout the paper, we find that corresponding to the density distribution given by equation (72) the gravitational potential at the point r, θ is

$$V = \theta_0 + \theta_1 P_1 + \theta_2 P_2 + \theta_3 P_3 + \dots \qquad (116),$$

where

$$\theta_s = \frac{4\pi}{2s+1} \left\{ \frac{1}{r^{n+1}} \int_{R_0}^{r} \rho_s r^{n+2}\, dr + r^n \int_{r}^{R_1} \frac{\rho_s\, dr}{r^{n-1}} \right\} \qquad (117).$$

The functions $\rho_1, \rho_2 \ldots$ are to be determined from the condition that V and ρ shall satisfy the three equations of equilibrium, which are of the form

$$\frac{1}{\rho}\frac{d\varpi}{dx} = \frac{dV}{dx}.$$

An Isothermal Nebula.

§ 40. Let us suppose, for the sake of simplicity, that the nebula is at uniform temperature, and extends from $r = 0$ to $r = \infty$. We have already seen (equation (77)) that the critical vibration for a nebula initially isothermal, is one in which the nebula remains isothermal. Hence it follows that if a nebula changes its configuration through coming to a point of bifurcation, when moving on a series of isothermal and spherical configurations, then the new series will also be one in which the equilibrium is isothermal.

We may now write $\varpi = \kappa\rho$, where κ is a constant, and the three equations of equilibrium become equivalent to the single equation,

$$\kappa \log \rho = V + c,$$

or

$$\rho = e^{\frac{V+c}{\kappa}} \qquad (118).$$

Now the series in question is, as we have seen, approximately represented, near to the point of bifurcation, by taking only two terms of (115), and consequently only two terms of (116). In this case equation (118) becomes:

$$\rho_0 + \rho_1 P_1 = e^{\frac{\theta_0 + c}{\kappa}} \left\{1 + \frac{\theta_1}{\kappa} P_1 + \tfrac{1}{2}\left(\frac{\theta_1}{\kappa} P_1\right)^2 + \ldots\right\} \qquad (119).$$

Equating coefficients, we find that ρ_0 is given by the equation

$$\rho_0 = e^{\frac{\theta_0+c}{\kappa}},$$

the same equation as in the case of perfect spherical symmetry. Also ρ_1 is given by the equation

$$\rho_1 = \frac{\rho_0}{\kappa}\theta_1.$$

It will be easily verified that this equation is exactly equivalent to our former equation (38). The equation contains an arbitrary multiplier in its solution. This may be taken to be a, the parameter of the series, so that we may write

$$\rho_1 = a\sigma_1,$$

where σ_1 is a completely determined function of r. Thus, as far as a, the solution is seen to be

$$\rho = \rho_0 + a\sigma_1 P_1.$$

We shall now show that, as far as a^2, the solution is

$$\rho = \rho_0 + a^2\sigma_{02} + a\sigma_1 P_1 + a^2\sigma_2 P_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (120).$$

The substitution of this in equation (118) leads to

$$\begin{aligned}\rho_0 + a^2\sigma_{02} + a\sigma_1 P_1 + a^2\sigma_2 P_2 \\ = e^{\frac{\theta_0+c}{\kappa}}\left\{1 + \frac{a\phi_1}{\kappa}P_1 + \tfrac{1}{2}\left(\frac{a\phi_1}{\kappa}P_1\right)^2 + \ldots + \frac{a^2\phi_2}{\kappa}P_2 + \ldots + \frac{a^2\phi_{02}}{\kappa} + \ldots\right\},\end{aligned}$$

where ϕ_1 stands in the same relation to σ_1 as does θ_1 to ρ_1. The right-hand member of this equation is equal to

$$e^{\frac{\theta_0+c}{\kappa}}\left\{1 + \tfrac{1}{6}\frac{a^2\phi_1^2}{\kappa^2} + \frac{a^2\phi_{02}}{\kappa} + \frac{a\phi_1}{\kappa}P_1 + \left(\frac{a^2\phi_2}{\kappa} + \tfrac{1}{3}\frac{a^2\phi_1^2}{\kappa^2}\right)P_2 + \ldots\right\},$$

in which the unwritten terms are of degree at least equal to 3 in a.

Neglecting a^3 the equation is satisfied if

$$\rho_0 = e^{\frac{\theta_0+c}{\kappa}}, \quad \sigma_{02} = \rho_0\left\{\tfrac{1}{6}\frac{\phi_1^2}{\kappa^2} + \frac{\phi_{02}}{\kappa}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (121),$$

$$\sigma_1 = \frac{\rho_0\phi_1}{\kappa} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (122),$$

$$\sigma_2 = \frac{\rho_0\phi_2}{\kappa} + \tfrac{1}{3}\frac{\rho_0\phi_1^2}{\kappa^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (123).$$

These equations determine σ_{02} and σ_2 uniquely.

It is obvious that this method is capable of indefinite extension, and that the general form of configuration in the series will be given by

$$\begin{aligned}\rho = \rho_0 + a^2\sigma_{02} + a^4\sigma_{04} + \ldots + (a\sigma_1 + a^3\sigma_{13} + a^5\sigma_{15} + \ldots)\,P_1 \\ + (a^2\sigma_2 + a^4\sigma_{24} + \ldots)\,P_2 + (a^3\sigma_3 + a^5\sigma_{35} + \ldots)\,P_3 + \&c. \quad . \quad (124).\end{aligned}$$

§ 41. Let us examine in greater detail the solution as far as a^2, this being given by equation (115). The important question, as will be seen later, is the determination of the sign of σ_2. We therefore pass at once to the consideration of equation (123). Written out in full, this becomes

$$\sigma_2 = \frac{4\pi\rho_0}{5\kappa}\left\{\frac{1}{r^3}\int_0^r \sigma_2' r'^4\,dr' + r^2\int_0^r \frac{\sigma_2'}{r'}\,dr'\right\} + \tfrac{1}{3}\frac{\rho_0\phi_1^2}{\kappa^2}.$$

This equation may be transformed in the same way as equation (39). If we write

$$y = \frac{\kappa r}{\rho_0}\left(\sigma_2 - \frac{1}{3}\frac{\rho_0 \phi_1^2}{\kappa^2}\right),$$

we find that the above equation is equivalent to (*cf.* equations (47), (48), (49), (54), (56))

$$\frac{d^2 y}{dr^2} - \frac{6y}{r^2} = -4\pi\sigma_2 r = -\frac{4\pi\rho_0}{\kappa}\left\{y - \frac{r\phi_1^2}{3\kappa}\right\} \quad . \quad . \quad . \quad . \quad . \quad (125),$$

together with the two equations

$$\left[\frac{1}{r^2}\frac{d}{dr}(yr^2)\right]_{r=\infty} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (126),$$

$$\left[r^3\frac{d}{dr}(yr^{-3})\right]_{r=0} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (127).$$

Writing

$$u = \frac{2\pi\rho_0 r^2}{\kappa},$$

equation (119) becomes

$$\frac{d^2 y}{dr^2} - \frac{y}{r^2}(6 - 2u) = -\frac{r\rho_0\phi_1^2}{3\kappa^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (128).$$

Referring to the table of values for u, which will be found on p. 15 of Professor DARWIN'S paper, it appears that u increases from a zero value at the origin up to a maximum value of about 1·66; it then decreases to a minimum of about ·8, and after this increases to 1, its value at infinity. Thus the factor $6 - 2u$ has a range of values from 6 to about $2\frac{2}{3}$.

Now the solution of

$$\frac{d^2 y}{dr^2} - \frac{y}{r^2}n(n+1) = -\frac{r\rho_0\phi_1^2}{3\kappa^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (129)$$

is easily found to be

$$y = \frac{4\pi r}{2n+1}\left\{\frac{1}{r^{n+1}}\int_0^r \frac{\rho_0\phi_1^2}{3\kappa^2} r'^{n+2}\,dr' + r^n\int_r^\infty \frac{\rho_0\phi_1^2}{3\kappa^2 r'^{n-1}}\,dr'\right\} + C_1 r^{-n} + C_2 r^{n+1} \quad . \quad (130),$$

in which C_1 and C_2 are constants of integration, which may at once be put equal to zero, if n is positive, and if y is to satisfy conditions (126) and (127).

Comparing (128) with (129), we see that if u had a constant value u_0 at every point of the nebula, the value of y would be given by equation (130), in which C_1, C_2 would be put equal to zero, and n would be the positive root of

$$n(n+1) = 6 - 2u_0,$$

provided only that $6 - 2u_0$ were positive.

For the range of values for u_0 from $u_0 = 0$ to $u_0 = 1{\cdot}66$, the value of n would have a range of values from 2 to 1·2. Thus the form of solution is materially the same for all of these values of u_0. It will be seen without difficulty that the solution of (128), in which u has not a constant value, but varies over the range from 0 to 1·66 as r varies, will be such that the graph expressing y as a function of r will present the same features as are common to the graphs given by equation (130) for ranges of n from 2 to 1·2.

Now the value of y given by equation (130) is positive for all values of r, hence we infer that the solution of (128) is such that y is positive for all values of r. We therefore have, for all values of r,

$$\sigma_2 = \tfrac{1}{3}\frac{\rho_0\phi_1^2}{\kappa^2} + \text{a positive quantity},$$

so that σ_2 is positive for all values of r.

§ 42. We therefore see that the initial motion, in which u and Δ are each proportional to the first harmonic, will first break down owing to the introduction of terms involving the second harmonic. The sign of these terms is such that there is, in all the shells of which the nebula is composed, a diminution of density in the equatorial regions, and a condensation at both poles, which must be added to that given by the terms involving the first harmonic.

The nature of this motion will become clearer upon a reference to fig. 3. This figure consists of the four curves*

$$r = a_0 \qquad r = a_0 + a_1 P_1$$

$$r = a_0 + a_{11}P_1 + a_2 P_2 \qquad r = a_0 + a'_{11}P_1 + a'_2 P_2,$$

and these may be supposed to represent curves of equal density in the three stages. It is easy to see that of the pear-shaped surfaces of equal density, the equations of which contain the two first harmonics, some will be turned in one direction, and some in the other. For if they were all turned in the same direction the centre of gravity could no longer remain at the centre of co-ordinates. Thus, if the narrow ends of these pear-shaped figures point in one direction at infinity, we must, as we go inwards, come to a place at which they have the transition shape, namely, ellipsoids of revolution, and after this they will point in the opposite direction.

It appears, therefore, that the initial motion is such as to suggest the ultimate division of the nebula into two parts, this division being effected by the outer layers condensing about one radius of the nebula, so as to leave room for the ejection of a

* The particular values for which the curves are drawn are in the ratio $a_0 = 11$, $a_1 = 2$, $a_{11} = 5$, $a_2 = 2$, $a'_{11} = 7$, $a'_2 = 4$. Thus the equation of the last curves are in polar co-ordinates,

$$r = \frac{a_0}{11}(10 + 5\cos\theta + 3\cos^2\theta), \quad r = \frac{a_0}{11}(9 + 7\cos\theta + 6\cos^2\theta).$$

central nucleus in the direction of the opposite radius. Whether or not actual separation takes place would probably depend on the amount of the angular velocity.

It is of interest to compare the result just arrived at, with the corresponding result found by POINCARÉ for the motion when an ellipsoid of JACOBI first becomes unstable.* This is described as follows :—

"La plus grande portion de la matière semble se rapprocher de la forme sphérique, tandis que la plus petite portion de cette même matière sort de l'ellipsoïde par l'extremité du grand axe, comme si elle voulait se séparer de la masse principale."

Thus, although the initial motions are, since they start from different configurations, necessarily different, yet it would seem as if the final result was very much the same in the two cases. In either case we have a diminution of matter in the equatorial regions, suggesting the ultimate division of the mass into two, and in each case these

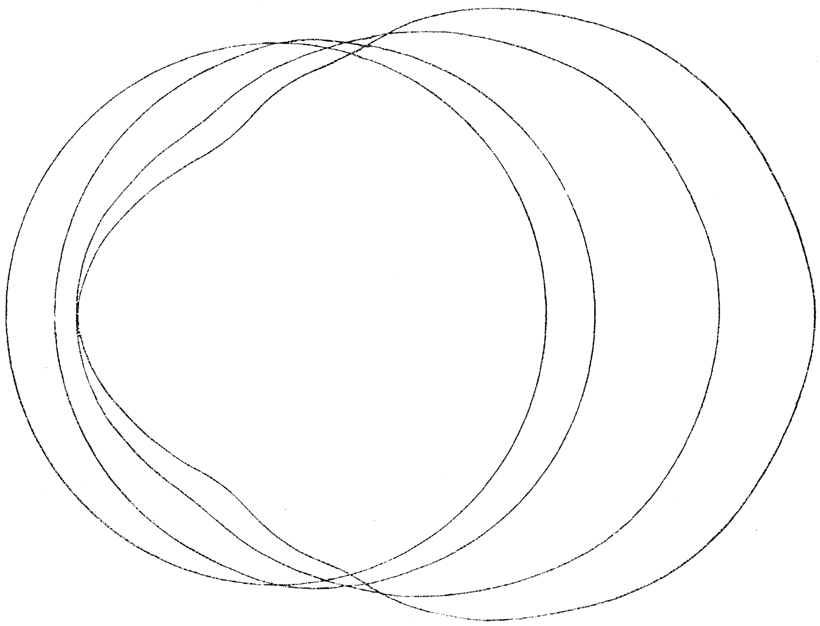

Fig. 3.

two masses are of unequal size, a result which could hardly have been foreseen without analysis.

§ 43. If the rate of cooling of a nebula is appreciable, the motion will not be along a "series" of equilibrium configurations. The value of p, the frequency which is nearest to instability, will be changing at a finite rate, and may run to some distance beyond the zero value, before the deviation of the nebula from the spherical shape is sufficient to invalidate the analysis of our paper. In this case we can imagine the first unstable vibration, that for which $p = 0$, being overtaken by other unstable vibrations of greater and greater frequency, the corresponding velocity of divergence

* 'Acta Mathematica,' vol. 7, p. 347.

from spherical symmetry becoming continually greater. It is therefore quite conceivable that the motion may become adiabatic at an early stage, and it is possible that it may be better imagined as a collapse or explosion, rather than as a gradual slipping from a spherical state of equilibrium into and through a series of unsymmetrical states of equilibrium.

But an examination of the physical character of the motion will show that in this extreme case, as also in any intermediate case, the motion must be, in its essentials, the same as that which has been found for the other extreme case, namely, that of infinitely slow cooling and perfect thermal equilibrium. In the spherical state, the outermost layers of gas may be regarded as stretched out in opposition to their gravitational attractions, being maintained in this state by the elasticity of the gas. The balance between these two agencies (which is, speaking loosely, measured by the stability function, u_∞) must be supposed to be continually changing, and instability always results from the same cause, namely, that the elasticity of these outer layers becomes inadequate to resist the gravitational tendency to collapse. In every case the outer layers concentrate about a single radius of the nebula, the axis of harmonics ($\theta = 0$ in equation (72)) and so increase the pressure along this radius, while decreasing that along the opposite radius ($\theta = \pi$). This pressure acting upon the inner layers of gas and the core sets them in motion, and in this way we have the tendency to separation into two nebulæ.

A Nebula in "Isothermal-adiabatic" Equilibrium.

§ 44. A nebula which consists of an isothermal nucleus with a layer in convective equilibrium above it, is said to be in "isothermal-adiabatic" equilibrium. At the surface at which the law changes from the adiabatic to the isothermal, the quantities ϖ, T and ρ must all be continuous.

The isothermal part is capable of executing a vibration of frequency $p = 0$ while remaining in isothermal equilibrium throughout, provided the forces acting upon it from the adiabatic part are the same as would act if the adiabatic part were replaced by an isothermal part in such a way that the whole made up an infinite isothermal nebula. If the nebula is rotating, the amplitude of vibration of the infinite nebula will vanish at infinity proportionally to some inverse power of r, this power increasing with the rotation. For sufficiently large rotations, the vibrations may be regarded as inappreciable except over the original isothermal nucleus, so that the vibration is approximately unaltered when the outer layers are again replaced by layers in convective equilibrium.

We see, therefore, that an "isothermal-adiabatic" nebula may become unstable, for sufficiently large rotations, through a vibration of order $n = 1$. No attempt is made to obtain any numerical results. We can, however, follow up the subsequent motion in the same way as in the case of an isothermal nebula.

Over the part of the nebula which is in adiabatic equilibrium, the relation between density and pressure is

$$\varpi = c\rho^{\gamma},$$

where c is a constant, so that the equations of equilibrium become

$$c\gamma\rho^{\gamma-2}\frac{d\rho}{dx} = \frac{dV}{dx}, \text{ \&c.},$$

and are therefore equivalent to the single equation

$$\frac{c\gamma}{\gamma - 1}\rho^{\gamma-1} = V - V_0,$$

where V_0 is the potential of the outer boundary of the nebula. This takes the form

$$V - V_0 = \frac{c\gamma\rho_0^{\gamma-1}}{\gamma - 1}\left\{1 + (\gamma - 1)\frac{\rho_1}{\rho_0}P_1 + \frac{(\gamma-1)(\gamma-2)}{2}\left(\frac{\rho_1}{\rho_0}P_1\right)^2 + \ldots \right.$$
$$\left. + (\gamma - 1)\frac{\rho_2}{\rho_0}P_2 + \ldots\right\},$$

or,

$$\theta_0 - V_0 + \theta_1 P_1 + \theta_2 P_2 + \ldots = \frac{c\gamma\rho_0^{\gamma-1}}{\gamma - 1}\left\{1 + \frac{(\gamma-1)(\gamma-2)}{6}\left(\frac{\rho_1}{\rho_0}\right)^2\right.$$
$$\left. + (\gamma - 1)\frac{\rho_1}{\rho_0}P_1 + \left((\gamma - 1)\frac{\rho_2}{\rho_0} + \frac{(\gamma-1)(\gamma-2)}{3}\left(\frac{\rho_1}{\rho_0}\right)^2\right)P_2 + \ldots\right\}.$$

It is obvious that equation (124) again gives the general form of solution, and that, as far as a^2, the equations are (*cf.* equations 121, 123)

$$\theta_0 - V_0 = \frac{c\gamma\rho_0}{\gamma - 1} \quad \ldots \quad (131),$$

$$\phi_1 = c\gamma\rho_0^{\gamma-1}\left(\frac{\sigma_1}{\rho_0}\right) \quad \ldots \quad (132),$$

$$\phi_2 = c\gamma\rho_0^{\gamma-1}\left\{\frac{\sigma_2}{\rho_0} + \tfrac{1}{3}(\gamma - 2)\left(\frac{\sigma_1}{\rho_0}\right)^2\right\} \quad \ldots \quad (133),$$

$$\phi_{02} = \frac{c\gamma(\gamma - 2)}{6}\rho_0^{\gamma-1}\left(\frac{\sigma_1}{\rho_0}\right)^2 \quad \ldots \quad (134).$$

Writing κ for $c\gamma\rho_0^{\gamma-1}$, we see that equations (132) and (133) may be written

$$\sigma_1 = \frac{\rho_0\phi_1}{\kappa} \quad \ldots \quad (135).$$

$$\sigma_2 = \frac{\rho_0\phi_2}{\kappa} + \tfrac{1}{3}(2 - \gamma)\frac{\sigma^2}{\rho_0\kappa} \quad \ldots \quad (136).$$

These are equations similar to (122) and (123); the last term in (136) is different from the last term in (123), but both terms agree in being invariably positive. Hence it appears that the question of the sign of σ_2 turns, as in § 37, upon the sign of the factor $(6 - 2u)$. We can no longer actually evaluate this factor, as in § 37, but it seems to be safe to infer from analogy that it will be positive at every point, and this in turn shows that σ_2 must be positive at every point. Hence it appears probable that the motion will be that described in § 38.

Rotating Nebula.

§ 45. The equations of an unsymmetrical series starting from a symmetrical configuration in which there is a finite amount of rotation would be extremely complicated, and no attempt to handle them is made in this paper. The correction for a small rotation will clearly consist merely of an increase in the terms containing the second harmonic, so that the general shape of the curves will be similar to that of the last two curves of fig. 3.

Little difficulty will be experienced in imagining the shape of curves appropriate to larger rotations.

Problems of Cosmic Evolution.

Infinite Space filled with Matter.

§ 46. A limiting solution of the equations of equilibrium (corresponding to $A = \infty$, $B = \infty$ in equation (114)) gives a nebula in which the density is constant everywhere. This solution may be supposed to represent infinite space filled with matter distributed at random. If space has no boundary there is presumably no need to satisfy a boundary-equation at infinity, so that ρ may have any value; if, however, this equation must be satisfied the only solution is $\rho = 0$.

Let us consider the former case. Space is filled with a medium of mean density ρ and of mean temperature T. Since the space under consideration is infinite, we may measure linear distances on any scale we please, and, by taking this scale sufficiently great, we can cause all irregularities in density and temperature to disappear. We may, therefore, suppose at once that the density and temperature have the constant values ρ and T.

The equations of motion for small displacements referred to rectangular axes are, in the old notation (*cf.* § 6), since V_0 and ϖ_0 are constants,

$$\frac{d^2\xi}{dt^2} = \frac{dV'}{dx} - \frac{1}{\rho_0}\frac{d\varpi'}{dx}, \text{ \&c.}, \quad \ldots \ldots \quad (137),$$

or, operating with d/dx, d/dy, d/dz, and adding

$$\frac{d^2\Delta}{dt^2} = \nabla^2 V' - \frac{1}{\rho_0}\nabla^2\varpi' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (138).$$

Since V′ is the gravitational potential of a distribution of density $-\Delta\rho$ (*cf.* § 6), we have

$$\nabla^2 V' = 4\pi\Delta\rho, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (139),$$

while if we suppose, for the sake of simplicity, that the motion is adiabatic, so that the ratio of pressure to density changes at a constant rate κ, we have (*cf.* equation (3), p. 5)

$$\nabla^2\varpi' = \kappa\nabla^2\rho' = -\kappa\rho_0\nabla^2\Delta.$$

Hence equation (138) becomes

$$\frac{d^2\Delta}{dt^2} - 4\pi\rho\Delta - \kappa\nabla^2\Delta = 0 \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (140).$$

The simplest solution of this is of the form

$$\Delta = \frac{1}{r} e^{i(pt \pm qr)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (141),$$

where

$$q^2 = \frac{p^2 + 4\pi\rho}{\kappa} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (142),$$

and the general solution can be built up by superposition of such solutions.

Now solution (141) gives $\Delta = 0$ at infinity, provided q is real, and therefore provided $p^2 + 4\pi\rho$ is positive, a condition which admits of p being imaginary. There is therefore a possible motion, which consists of a concentration of matter about some point, the amount of this concentration vanishing at infinity, and the amount at any point increasing, in the initial stages, exponentially with the time.

We conclude, therefore, that a uniform distribution in space will be unstable, independently of the mean temperature or density of this distribution.*

The Evolution of Nebulæ.

§ 47. We can also see that a distribution of matter which is symmetrical about a single point will be equally unstable. For, if this distribution of matter were perfectly

* An interesting field of speculation is opened by regarding the stars themselves as molecules of a quasi-gas. If space were Euclidean and unbounded, there would be no objection to this procedure, and we should be led to the conclusion that the matter of the universe must become more and more concentrated in the course of time. If space is non-Euclidean, this concentration might reach a limit as soon as the coarsegrainedness of the structure attained a value so great that the distance between individual units became comparable with the radii of curvature of space. In any case, it may reach a limit as soon as an appreciable fraction of the space in question becomes occupied by matter.

homogeneous, the whole mass of matter would form a spherical nebula of literally infinite extent, and would therefore be in neutral equilibrium. The introduction of even the smallest irregularities into this structure is equivalent to the application of an external field of force. This, as has already been seen, will destroy the spherical symmetry, and it can easily be seen that the motion from spherical symmetry is such as to lead to a concentration of matter about points of maximum density.

It appears, therefore, that the configuration which will naturally be assumed by an infinite mass of matter in the gaseous or meteoritic state consists of a number of nebulæ (*i.e.*, clusters round points of maximum density). We may either suppose the outer regions of these nebulæ to overlap, each nebula satisfying the gas-equations by being of infinite extent, or we may suppose the nebulæ to be distinct and of finite size, the interstices being filled by meteorites or other matter, which by continual bombardment upon the surfaces of the nebulæ supply the pressure which is required at these surfaces by the equations of equilibrium.

§ 48. What, we may inquire, will determine the linear scale upon which these nebulæ are formed? Three quantities only can be concerned: γ the gravitational constant, ρ the mean density, and λT the mean elasticity. Now these quantities can combine in only one way so as to form a length, namely, through the expression

$$\sqrt{\frac{\lambda T}{\gamma \rho}},$$

of which the dimensions will be readily verified to be unity in length, and zero in mass and time. We conclude, then, that the distance between adjacent nebulæ will be comparable with the above expression.

Now the value of γ is 65×10^{-9}, and if we assume the primitive temperature to be comparable with 1000° (absolute) we may take $\lambda T = 10^9$ (corresponding accurately to an absolute temperature of 350° for air, 2800° for hydrogen). If we take the sun's diameter as a temporary unit of length, the earth's orbit is (roughly) of diameter 200. If we suppose the fixed stars to be at an average parallactic distance of 0·5″ apart, measured with respect to the earth's orbit, we find for their mean distance apart, about 4×10^7 sun's radii. The density of the sun being, in C.G.S. units, roughly equal to unity, we may, to the best of our knowledge, suppose the mean density of the primitive distribution of matter to be about $(4 \times 10^7)^{-3}$, or say 10^{-23}. Substituting these values for γ, λT and ρ, we find as the scale of length a quantity of the order of $10^{19 \cdot 5}$ centims. The distance which corresponds to a parallax of 0·5″ would be about $10^{18 \cdot 6}$ centims. It will therefore be seen that we are dealing with distances which are of the astronomical order of magnitude.

The Evolution of Planetary Systems.

§ 49. Let us now regard a single centre, together with the matter collected round it, as the spherical nebula which is the subject of discussion. On account of the way in which it has been formed, this nebula will, in general, be endowed with a certain amount of angular momentum. We have seen that a primitive nebula of this kind may be supposed, under certain conditions, to become unstable. We have also seen that the motion, when the nebula becomes unstable, is such as to strongly suggest the ejection of a satellite.

As a nebula cools the rotation increases, owing to the contraction of the nebula, and Ω also increases. Thus the quantity $\Omega^2/3\lambda^2T^2$, which measures the rotational tendency to instability, has a double cause of increase; firstly owing to the increase in Ω, and secondly owing to the decrease in T. We can accordingly imagine the primitive nebula becoming unstable time after time, throwing off a satellite each time.

In the usually accepted form of the nebular hypothesis, the rotation is supposed to be the sole cause of instability, so that the system resulting from a single nebula ought theoretically to be entirely symmetrical about an axis. On the view of the present paper, there is no reason for expecting this symmetry. For large rotations of the primitive nebula, the configuration of the resultant planetary system will approximate to perfect symmetry, but for small rotations, a slight irregularity occurring at the critical moment, at a point out of the equatorial plane, may produce a satellite of which the orbit is far removed from the equatorial plane.

In conclusion, two particular cases of "irregularities" may be referred to. If the nebula is penetrated by a wandering meteorite, at a moment at which it is close to a state of instability, the presence of the meteorite will constitute an irregularity, and may easily result in the formation of a satellite. And if a quasi-tide is raised in the nebula by the presence of a distant mass, the same result may be produced. In the former case, the plane of the satellite would, if the rotation is sufficiently small, be largely determined by the path of the meteorite; in the second case, by the position (or path) of the attracting mass. It would not, in either case, depend much upon the axis of rotation of the nebula.

CONCLUSION.

§ 50. To sum up, it appears that the behaviour of a gaseous nebula differs in at least two important respects from that of an incompressible liquid. In the first place, it differs as regards the amount of rotation which is required to produce instability, and, in the second place, it differs as regards the disposition of the orbits of the planets which will be formed out of the primitive nebula. It will be noticed that no definite numerical results have been obtained; my aim has been to obtain qualitative rather than quantitative results, so as to show, if possible, that the results to be expected for a gaseous nebula are of so much more general a kind than those usually inferred from the analogy of a liquid mass, that no difficulty need be experienced in referring existent planetary systems to a nebular or meteoritic origin, on the ground that the configurations of these systems are not such as could have originated out of a rotating mass of liquid.

In conclusion, I wish to express my indebtedness to Professor DARWIN for much assistance which I have received from him throughout the course of my work.