Project on oscillatory α^2 dynamos

Background

Textbook knowledge (e.g. Parker, 1979) suggests that $\alpha\Omega$ dynamos tend to be oscillatory while α^2 dynamos tend to be non-oscillatory. That this does not have to be true was clear for some time (Baryshnikova & Shukurov, 1987; Rädler & Bräuer, 1987), but only recently it became clear that such dynamos can also have migratory wave solutions similar to what is seen in the Sun (Mitra et al., 2010; Masada & Sano, 2014). The goal of this project is to present a "textbook example" of an analytically solvable α^2 dynamo in 1-D. Particular aims of this project include a better understanding of

- the excitation condition,
- the migration properties, and the
- dependence on boundary conditions.

Project details

The one-dimensional α^2 dynamo is governed by the equations

$$\begin{aligned}(\partial_t - \partial_z^2)A_x + \alpha\partial_z A_y = 0,\\ (\partial_t - \partial_z^2)A_y - \alpha\partial_z A_x = 0,\end{aligned} \tag{1}$$

with four boundary conditions

$$\begin{aligned}A_x = A_y = 0 \qquad & \text{on } z = 0,\\ \partial_z A_x = \partial_z A_y = 0 \qquad & \text{on } z = \pi/2.\end{aligned} \tag{2}$$

1. Show that Equations (1) and (1) can be combined into a fourth order equation that is obeyed both by A_x and by A_y, so we drop in the following subscripts x and y and write the equation in the form

$$\left[(\partial_t - \partial_z^2)^2 + \alpha^2\partial_z^2\right] A = 0, \tag{3}$$

   with four boundary conditions

$$\begin{aligned}A = (\partial_z^2 + \alpha^2)\partial_z A = 0 \qquad & \text{on } z = 0,\\ \partial_z A = (\partial_t - \partial_z^2)A = 0 \qquad & \text{on } z = \pi/2.\end{aligned} \tag{4}$$

2. Seek the solution in the form

$$A = \sum_{i=0}^{4} C_i e^{q_i z} e^{-\omega t}, \tag{5}$$

   where the q_i obey

$$\left[(\mathrm{i}\omega + q_i^2)^2 + \alpha^2 q_i^2\right] A = 0. \tag{6}$$

3. Show that this leads to the dispersion relation

$$q_i^4 + (2\mathrm{i}\omega + \alpha^2) q_i^2 - \omega^2 = 0 \tag{7}$$

with the solutions

$$q_i = \pm \left[-(\mathrm{i}\omega + \alpha^2/2) \pm (\mathrm{i}\omega\alpha^2 + \alpha^4/4)^{1/2} \right]^{1/2}, \tag{8}$$

where $i = 1, 2, 3, 4$ correspond to the signs $++, +-, -+, --$ in Equation (8).

4. Argue that the four boundary conditions can be satisfied by solving the following eigenvalue problem $\mathsf{M}\boldsymbol{C} = 0$ with $\boldsymbol{C} = (C_1, C_2, C_3, C_4)^T$ and

$$\mathsf{M} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ (q_1^2+\alpha^2)q_1 & (q_2^2+\alpha^2)q_2 & (q_3^2+\alpha^2)q_3 & (q_4^2+\alpha^2)q_4 \\ (\mathrm{i}\omega+q_1^2)\,e^{\tilde{q}_1} & (\mathrm{i}\omega+q_2^2)\,e^{\tilde{q}_2} & (\mathrm{i}\omega+q_3^2)\,e^{\tilde{q}_3} & (\mathrm{i}\omega+q_4^2)\,e^{\tilde{q}_4} \\ q_1 e^{\tilde{q}_1} & q_2 e^{\tilde{q}_2} & q_3 e^{\tilde{q}_3} & q_4 e^{\tilde{q}_4} \end{pmatrix}. \tag{9}$$

where we have defined $\tilde{q}_i \equiv q_i \pi/2$ as a shorthand.

5. Find solutions and compare with numerical solutions using, for example, the setup prepared in the Pencil Code exercises; `http://www.nordita.org/~brandenb/teach/PencilCode/MeanFieldCartesian_1D.html`

References

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Mitra, D., Tavakol, R., Käpylä, P. J., & Brandenburg, A., "Oscillatory migrating magnetic fields in helical turbulence in spherical domains," *Astrophys. J. Lett.* **719**, L1-L4 (2010).

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