ASTR/ATOC-5410: Fluid Instabilities, Waves, and Turbulence

Project description November 18, 2016, Axel Brandenburg

Project on stratified shear flows

Background

The stabilizing buoyancy force due to shear is an important aspect to shear-flow instabilities and is particularly relevant in meteorology. Taylor, G. I. (1931) and Goldstein (1931) published the first work on this problem in the same issue of Proc. Roy. Soc.

In Handout 9 on stratified shear flows we discussed the Goldstein-Taylor equation as a straightforward generalization of Rayleigh's instability equation. We arrived at the equation

$$c_\mathrm{i} \int \frac{N^2 - \frac{1}{4}(U')^2}{|U-c|^2} |\psi|^2 = -c_\mathrm{i} \int \left(|\psi'| + k^2 |\psi|^2 \right) \mathrm{d}x \tag{1}$$

with $c = c_\mathrm{r} + \mathrm{i}c_\mathrm{i}$. This shows that the flow cannot be unstable if $N^2 - \frac{1}{4}(U')^2 > 0$ everywhere in the domain. In terms of the *gradient Richardson number* $\mathrm{Ri}(z) = N^2/(U')^2$, this means $\mathrm{Ri} > 1/4$ for stability.

Project details

The idea of the project is to solve this equation numerically. To solve the fully nonlinear problem, we can use a hydrodynamics code such as ATHENA[1], DEDALUS[2], or the PENCIL CODE[3]. The Rayleigh instability problem is a limiting case ($\nu \to 0$) of the Kelvin-Helmholtz instability, which is used as a common test problem for numerical codes. It has been used last Summer during the Bootcamp for Computational Fluid Dynamics`http://www.nordita.org/~brandenb/teach/PencilCode/LCDworkshop2016/` at the Laboratory for Computational Dynamics (LCD), which is upstairs in the third floor. See Lecoanet et al. (2016) for details, but in the absence of stratification.

1. Review the literature on numerical studies of the stratified shear-flow problem.
2. Start by solving the usual Kelvin-Helmholtz instability and check the validity of the inflection-point theorem and the more stringent Fjørtoft (1950) theorem.
3. Add constant gravity to the problem. Verify that the resulting stratification obeys the hydrostatic solution. You may consider either the fully compressible case (e.g., with an isothermal equation of state) or the incompressible Boussinesq problem.
4. Check numerically the validity of the Richardson criterion, which says that the flow cannot be unstable if $N^2 - \frac{1}{4}(U')^2 > 0$ everywhere in the domain. Here, N is the buoyancy frequency. In terms of the *gradient Richardson number* $\mathrm{Ri}(z) = N^2/(U')^2$, this means $\mathrm{Ri} > 1/4$ for stability. The simplest case is that on an isothermal layer. Use $\boldsymbol{g} = -\hat{\boldsymbol{y}}g$ with $g > 0$ and show that in that case,
$$\boldsymbol{\nabla} \ln \rho = \boldsymbol{g}/\gamma c_\mathrm{s}^2 = -\hat{\boldsymbol{y}}/H_\rho, \quad \boldsymbol{\nabla} s/c_\mathrm{p} = +\hat{\boldsymbol{y}}\,(\gamma-1)/H_\rho. \tag{2}$$

[1] `http://www.astro.princeton.edu/~jstone/athena.html`
[2] `http://dedalus-project.org/`
[3] `http://pencil-code.nordita.org/`

If you choose to work with the Pencil Code, you may start with a run that has been prepared in `http://lcd-www.colorado.edu/~axbr9098/teach/PencilCode/material/KelvinHelmholtz/gravity/reference_run/`.

5. Consider the same problem in the presence of turbulence. Here, the turbulence can be driven by an external monochromatic body force (forcing wavenumber k_{f}). However, you may also consider the problem of turbulence as a result of the shear-flow instability itself.

6. Review your results in view of recent discussions of the validity of Richardson criterion in the literature; see Zilitinkevich et al. (2007) who claim that "The proposed model permits the existence of turbulence at any gradient Richardson number".

References

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